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CONTENTS

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THE MATHEMATICS TEACHER

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What Price Factoring?

By William David Reeve
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One of the hardest things to understand is why teachers and authors of textbooks in this modern age continue to overemphasize the importance of factoring in the senior high school. Of course, the student needs to know how to factor in order to put formulas in better shape for computation, but why should we continue to overemphasize its importance by giving a lot of unnecessary examples in formal factoring. Why factor such formal cases as x^2+5x+6, unless such work can be used later in the course? Some college teachers say, "The high school teachers should teach their students how to factor such quadratic functions as x^2+5x+6 because they will need it when they come to study the calculus." However, there are two important reactions one can make about such a statement.

1. Many high school students will never go to college.

2. Those that go to college can learn it there if, and when, they ever need to know it.

The day is past when the college teacher can dictate the high school curriculum in mathematics. If he wants his students to know certain types of factoring, the college teacher should teach it to them. The responsibility is his. When it is necessary, the student will understand the nature and practical value of algebraic formulas and the idea of factoring better, when he realizes how certain formulas result from multiplying one factor by another to obtain a certain result.

In multiplication the factors are given and the result is required. For example, if we multiply $a+b$ by x, we get $ax+bx$ as a product. Similarly, if we multiply $a+b$ by $a-b$, we get a^2-b^2 as a product. Such products are called *special products.* In division the result and one factor are given and the other factor is to be found. Thus, if we divide $ax+bx$ by x, we get $a+b$. Similarly, if we divide a^2-b^2 by $a+b$, we get $a-b$ and so on. If the result is given and the factors are required, the process of finding the factors is called *factoring.* Thus, $a+b$ and $a-b$ are the factors of a^2-b^2, but why should the students be required to know them unless he can use them in his later work?

All such work as traditionally presented in the high school is quite formal. Why should anyone know how to factor? Moreover, if factoring is necessary, what forms of factoring are most important and where should they be taught?

The principal use of factoring as indicated above is to enable the student to put a formula in better shape for computation. Factoring is also useful in applied problems where the solution of equations is involved or where one is required to transform certain algebraic functions. To find factors in such cases and

to employ them in solving problems is identical in arithmetic and algebra.

In order to better understand some of the realistic uses of factoring the following points may be helpful:

In the ninth grade there is no longer any question as to the importance of three types of factoring. They are:

1. Factoring out a common monomial factor for which the traditional type form is $ax+bx+cx=x(a+b+c)$.

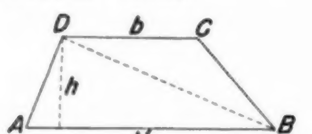
2. Factoring the difference of two squares for which the type form is $a^2-b^2=(a+b)(a-b)$.

3. Factoring a perfect trinomial square for which the type form is

$$a^2 \pm 2ab+b^2=(a \pm b)^2.$$

These three types have geometric applications that are genuine. This very fact helps to reduce the formalism of factoring as it has been traditionally taught. No one wants to eliminate these three types from the course of study in mathematics at any point. Now let us look at the opportunity the teacher has to emphasize the reality of these three cases.

Illustrative example 1. Develop a formula for the area of a trapezoid $ABCD$ shown here where h is the height and b and b' are the upper and lower bases respectively. Find the area if $h=5.2$, $b=9.4$ and $b'=12.8$.



How to find the area of a Trapezoid

Solution. Draw the diagonal BD forming triangles ABD and BCD. Then the area of the trapezoid can be found by first finding the areas of these triangles and then adding the results.

Thus,
Area of $\triangle BCD=\frac{1}{2}hb$
Area of $\triangle ABD=\frac{1}{2}hb'$

Therefore, the area of the trapezoid is given by the formula

$$A=\tfrac{1}{2}hb+\tfrac{1}{2}hb'.$$

Now to compute the area of the trapezoid we must substitute the given values in the formula. This is called *evaluating* the formula. However, it is easier to evaluate this formula if we first factor out the common monomial factor $\frac{1}{2}h$ and get

$$A=\tfrac{1}{2}h(b+b').$$

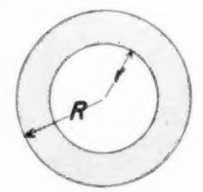
Substituting $h=5.2$, $b=9.4$ and $b'=12.8$ in the formula

$$\begin{aligned} A &=\tfrac{1}{2}\times 5.2(9.4+12.8) \\ &=\tfrac{1}{2}\times 5.2\times 22.2 \\ &=57.72. \end{aligned}$$

Here again we have a special product

$$\tfrac{1}{2}h(b+b')=\tfrac{1}{2}hb+\tfrac{1}{2}hb'.$$

Illustrative example 2. Develop a formula for the area of a ring (annulus) like the one shown here where R is the radius of the larger circle and r is the radius of the smaller circle. Find the area of the ring if $R=6.3$ and $r=2.4$.



How to Find the Area of a Ring

Solution. The area of the ring can be found by subtracting the area of the smaller circle πr^2 from the area of the larger circle πR^2. Thus

$$A=\pi R^2-\pi r^2.$$

To compute the area we must substitute the given values of R and r in this formula. Again, it will be easier to evaluate the formula if we first factor the right member by taking out the common monomial factor π.

Thus,

$$A=\pi(R^2-r^2).$$

The second factor in the right member is in the form of the difference of two squares and can be further factored into $R+r$ and $R-r$. Finally, then

$$A=\pi(R+r)(R-r).$$

Substituting 6.3 for R and 2.4 for r in this formula we get

$$\begin{aligned} A &=\pi(6.3+2.4)(6.3-2.4) \\ &=\pi\times 8.7\times 3.9 \\ &=33.93\pi. \end{aligned}$$

In this example $(R+r)(R-r)=R^2-r^2$ is also a special product. Similar applications can be made of the third type.

The preceding example shows how the special product $R^2-r^2=(R+r)(R-r)$ provides us with a method of making calculations easier. In fact, the expressions linked by the sign of equality in $R^2-r^2=(R+r)(R-r)$ simply represent two different ways of carrying out the same calculation, of which the one on the right is the easier.

Knowledge of special products also en-

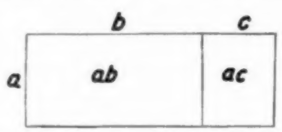
ables us to multiply certain arithmetic numbers with great speed. Thus, the product of 52 by 48 may be written

$$(50+2)(50-2)=50^2-2^2$$
$$=2496$$

Again the teacher should show his students how informal geometry can be used to put meaning and insight into certain special products.

According to research studies that have been made, one of the most common errors in the work of students in elementary algebra is the failure to multiply the second term of a binomial by the monomial. Thus, in multiplying $b+c$ by a which is about the simplest known type product the students wrote $ab+c$ which of course is incorrect.

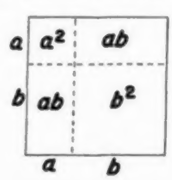
In the figure shown here the area of the rectangle is seen to be made up of two smaller rectangles ab and ac showing clearly that when $b+c$ is the length and



a is the width, the area is $ab+ac$ not $ab+c$. When the student makes the error of writing "$a(b+c)=ab+c$" the teacher should ask him to illustrate the algebra by means of a geometric drawing.

Another common error in algebra is to say that "$(a+b)^2=a^2+b^2$" which is obviously incorrect. If we draw a geometric figure to represent the square on a line segment $a+b$ units long, we can readily see that the square on $a+b$ is made up of two squares a^2 and b^2 and two rectangles each having an area ab. Thus

$$(a+b)^2=a^2+2ab+b^2.$$

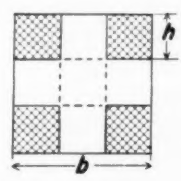


None of the examples above are formal in the sense that most factoring examples in current textbooks are. Surely students would see more reason for factoring if the subject matter were more realistic as in the examples above. If this were true, the injection of some formal examples of the various factoring types would be acceptable for purposes of drill so as to fasten the students' ideas of the various types.

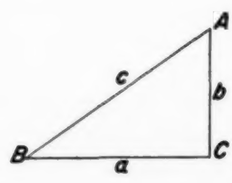
We all know that many students like to factor the very formal examples, even though they do not understand what they mean. This is because the examples are not difficult. However, the ease with which tasks are done is no criterion for teaching them. Students also like to solve first degree equations in two unknowns, because they are easy, but it is very difficult to find realistic cases where the need for solving such equations seems urgent as every mathematics teacher knows.

Here are some problems which illustrate how some of the newer books are trying to help the student to get a better understanding of some of the necessary types of factoring:

1. Open square metal boxes are often constructed by cutting out four equal squares from the corners of a square piece of metal and folding the four flaps of metal up as shown in the figure. Write a formula for the area (A) of the inner surface of the box, rewrite the formula in factored form and find A when $b=8.4$ in. and $h=2.7$ in.



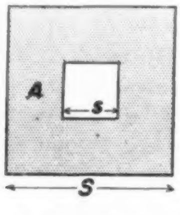
2. The Pythagorean formula for the right triangle shown here is $c^2=a^2+b^2$ where c is the hypotenuse and a and b are the two legs. Write a formula for a in terms of b and c, factor the other member and find the value of a when $c=18$ and $b=12$.



3. Show that the shaded area A in the figure shown here is given by the formula

$$A=(S+s)(S-s)$$

where S is a side of the large square and s is a side of the small square. Find the value of A if $S=20$ and $s=7$.



4. A carpet 30 ft. square is to be placed in a room 35 ft. square. Find the area of the strip and the cost of painting this area at $1.10 per square yard. Write a formula to be used in calculating the approximate cost of painting similar strips at c cents per square yard, the carpet to be f feet square and the room r feet square.

There is a fourth type of factoring which is included in most of the ninth grade algebras for which no affirmative case can be made. I refer to the factoring of the general trinomial ax^2+bx+c or at the least the special trinomial, x^2+px+q. This type should be omitted for the following reasons:

It is an obsolete topic for ninth-graders because no real use can be made of it in applied problems in simplifying formulas as can be done in the three cases already discussed.

Some people may argue that this type of factoring should be taught in order that the students may know how to solve certain quadratic equations that they may later meet in life situations. But, since it is impossible to find a quadratic equation in a real life situation that can be solved by factoring, what is the reason for including this type in the ninth grade? Moreover, no extramural board today requires this type of factoring. The New York Regents no longer gives an examination in ninth grade algebra. Why then all the attention to this type of factoring? Only pure quadratics like $A=\pi r^2$ and $s=16t^2$ are now required by extramural examining boards. Even in Congdon's study on the mathematics that is necessary to understand college physics and chemistry he found not a single problem in all of Stewart's physics (a text used at Columbia College) that required a quadratic equation for its solution. The kind of quadratic equation that was used in the construction of the Holland Tunnel under the Hudson River in New York cannot be solved by factoring. It can only be solved by the method of completing the square or by the formula. Thus, the only way this fourth type of factoring can be used in the ninth grade algebra books is to ask the students to solve artificial quadratic equations like $x^2+5x+6=0$ which the textbook writer "cooks up" by multiplying $x+2$ by $x+3$ and also to simplify formal exercises in fractions that involve quadratics that can be factored. Thus, the vicious circle is completed. We introduce this type of factoring so that we can use it in solving fractions which are "cooked up purposely" so that we can use the type of factoring which came earlier in the course!

We should substitute more important topics for this fourth type of factoring in the ninth grade like some of the pure quadratic equations as they occur in well known formulas in geometry and elsewhere. It is only in this way that we can reduce the formalism which Hedrick denounced all his life.

Factoring the general quadratic has little value even in the eleventh grade, but perhaps it could be mentioned there.

It is gratifying to see that some of the authors of intermediate algebras are beginning to realize the artificiality of solving quadratic equations by factoring. One recent textbook in intermediate algebra first discusses the solution of incomplete quadratics, then complete quadratics by the methods of completing the square and the formula, and leaves the factoring method until last. Then the authors say that some quadratic equations can be solved by factoring, that this method of solving them is simple, but that their treatment has been put last because the method can be applied to some but not all quadratic equations. Then they go

on to point out that "factoring is seldom possible with equations derived from real problems in science, engineering, and industry." In such cases only the method of completing the square or the formula method can be used.

When I talk about the use of factoring I am not thinking of usefulness or practicality in a narrow sense. I realize that sometime, somewhere, some students may "get a kick" out of factoring a function like "$6x^2-x-12$ in the domain of integers." If some teachers want to teach certain students to factor functions of that type, let them do it for fun as some people do cross word puzzles. However, like cross word puzzles this type of factoring leads nowhere. It is suitable material for the *algebra museum* in so far as ninth grade algebra is concerned.

Heller in an important study "Concerning the Development of Factoring in Textbooks of Elementary Algebra Published in America and England from 1631 to 1890" has convinced me of what I long suspected, that countless operations have been introduced into elementary algebra textbooks for which practical mathematics had no need, and that forms of factoring were stressed which could not be justified except by some untenable form of mental discipline theory. Obviously, I cannot here produce the evidence contained in the above study to bring out my point, but it is available here for those who wish to read it. From Van Schooten (who started the trouble) on down to Wentworth, one author after another copied from his predecessors. Wentworth said in 1906:

> The chapter on factors has been made full in order to shorten subsequent work. The easy methods of resolving trinomials into factors, and the explanation of the Factor Theorem will be found of great service, in abridging many algebraic processes. Examples showing short methods of finding highest common factor of compound expressions, and of solving quadratic equations by resolution into factors, should receive special attention when reached.

Now, who believes all that is true for a modern ninth-grade class in algebra? I now quote briefly from pages 100–101 of the above study by Heller:

> In summary it may be noted that seventeenth century activity in factoring centered about crude attempts to solve equations of degree higher than the second, and upon the reduction to lowest terms of fractions which appeared in algebra textbooks. During the eighteenth century the topic came in for slight mention.
>
> Factoring developed to a place of importance during the nineteenth century in textbooks in America and England. Its progress was slow at first, but with the appearance of Wentworth's books the popularity of the topic suddenly increased. Insofar as nineteenth century authors defended the emphasis upon the topic, they did so on the basis of its usefulness to shorten manipulative work.
>
> The period in which factoring became important coincided with the period in which manipulation of fractions received increasing emphasis. During these years, the manipulation of fractions with monomial terms in textbooks yielded place to fractions with factorable polynomials as terms. It was during these years also that the subject of arithmetic was emphasizing rapid calculation, and arithmetic books were stressing cancellation devices.
>
> Insofar as applications within the textbooks were concerned, the applications of factoring were largely limited to the exercises with fractions. It was, for instance, forty years after quadratic trinomials were first introduced in chapters on factoring before the solution of quadratic equations by factoring became a common practice in the textbooks.

These quotations are good samples of the sort of buttressing evidence one can obtain from this study against the continued teaching of formal factoring in the elementary parts of algebra.

In the tenth grade one new type of factoring will have to be taught if a proof is given for Hero's formula, but, since not all of the geometry textbooks include this proof today, we may not need any new factoring types in the geometry course at all. This type of factoring is not new in a sense anyway because it involves the difference of two squares, but in making the proof it is necesary to set up the material so that it will be in the form of the difference of two squares before it can be factored. The proof is really quite difficult as all teachers of geometry know, but if the proof is required the type of factoring involved can be justified.

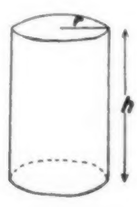
In the intermediate algebra course, and

in trigonometry, factoring should be reviewed and extended, but even here great care should be exercised to see that the work is not too mechanical. This can be done if the teacher brings in more applications of factoring in simplifying formulas and fractions that occur in geometry and elsewhere. For example, the teacher might assign examples in multiplication as follows:

1. $p(1+rt)$	4. $\pi l(R^2-r^2)$
2. $2\pi r(r+h)$	5. $\pi(R^2-4r^2)$
3. $\frac{1}{2}(a+b+c)$	6. $t^2(2gt-k)$

We have already seen how factoring can be used to simplify some of the real problems one may meet in his daily life whether inside or outside of school. Further practice with such cases should help the student to become better equipped to understand and carry on such computations.

Illustrative example 1. The formula for the total surface (S) of a right circular cylinder shown in this figure is $S=2\pi r+2\pi rh$ where r is the radius of the base and h is the height. Write this formula in factored form and find out what the total surface S would be if $r=3.8$ in. and $h=5.2$ in.



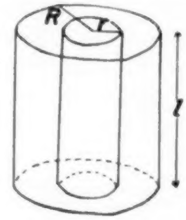
Solution. The monomial factor $2\pi r$ is common to each term of the right member of the above formula; so we may say that

$$S=2\pi r(r+h).$$

Substituting the given values of r and h in the right member

$$S=2\pi\times 3.8(3.8+5.2)$$
$$=2\pi\times 3.8\times 9$$
$$=68.4\pi \text{ in.}$$

Illustrative example 2. The formula for the volume a pipe like the one shown here is $V=\pi lR^2-\pi lr^2$ where l is the length of the pipe, R is the outside radius and r is the inside radius. Write this formula in factored form and find V if $l=20$ in., $R=\frac{5}{8}$ in. and $r=\frac{1}{2}$ in.



Solution. By taking out the common monomial factor πl from the right member of the formula we have

$$V=\pi l(R^2-r^2).$$

Now we know from previous work that

$$R^2-r^2=(R+r)(R-r)$$

where R^2-r^2 represents the difference of two squares and $R+r$ and $R-r$ are the factors. Hence

$$V=\pi l(R+r)(R-r)$$

and substituting for values of l, R and r,

$$V=\pi\cdot 20\cdot\tfrac{9}{8}\cdot\tfrac{1}{8}$$
$$=2.81\pi.$$

The teacher of mathematics can find other real examples like these to give his students for further practice.

Maybe we can now look forward to some improvement in the topic of factoring, but it requires a long time to take up the lag in mathematical education. Some of the reforms which the leaders in mathematics were suggesting twenty to thirty years ago are only now being adopted by the rank and file of classroom teachers in mathematics.

Cryptography as a Branch of Mathematics

By Richard V. Andree
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Much has been written concerning the advisability of providing "special courses of reduced content" for students with low ability or interest in mathematics. More has been written pondering the observation that the watered down version of algebra and/or geometry, so common at present, is *not* meeting the demand of either student or educator. Another subject which has worn out typewriter ribbons is the remark that better students should receive additional instruction in mathematics beyond their regular course work.

It is not the purpose of this paper to add to the general literature on any of these subjects, but rather to suggest one example which may serve teachers wishing to put the generalities mentioned above into practice. The field is cryptography—the study of codes and ciphers. It is a fascinating subject. The elementary portions are surprisingly easy. The problem of motivation, often so acute in "general mathematics," vanishes in the almost universal interest in secret communication. Poe's "Gold Bug," Doyle's "Adventure of the Dancing Men," and Sayers' "Have His Carcass" successfully play upon this interest. Why shouldn't we? I do *not* advocate an entire course in cryptography, but certainly one unit in a "general mathematics" course could well be devoted to cryptography. Logical reasoning, the testing of hypotheses by examining conclusions drawn from the data, and the limits of reliability in statistical data, are concepts which cease to be mere words, becoming real tools every time a cipher is broken (that is, solved and read without prior knowledge of the key).

One difficulty may well be the general lack of cryptographic knowledge on the part of mathematics teachers. This article suggests easily available sources for such information. The first distinction a teacher should make is between "code" and "cipher." A *cipher* is a means of secret communication in which the enciphered message (called "cipher") is as long as or longer than the original message (called plain text). This is what the comic strips usually refer to as "a code." A true *code* is quite different. It requires a code dictionary to write, and one five letter group may stand for an entire sentence or paragraph. The encoded message is shorter than the original. For example the code word CABLE may stand for, "We have almost completed the investigation you requested and have found some reason to doubt the loyalty of the man (woman) in question. We suggest you take no action until our file is complete. Our final report should be in your hands about ________." The next code word would indicate the date and possibly the time as well. Thus two short code words convey an entire paragraph of plain text. Code has the advantage of brevity, but it has the disadvantage that a code book must be carried by both sender and receiver and the vocabulary of the code is necessarily limited. Codes can be broken (read without a key) if sufficient material is on hand, but it is often easier to steal and photograph the code dictionary.

Ciphers are divided into two general types:

Transposition cipher in which the plain text letters are unchanged but their order is scrambled in some systematic way.

Substitution cipher in which letters (or groups of letters) are substituted for letters (or groups of letters) of the plain text.

A transposition cipher is easily constructed by inscribing the message by rows in a rectangle and transcribing it by columns. Nulls are added as needed.

Plain text: Meet us at eight thirty.

M	E	E	T	U
S	A	T	E	I
G	H	T	T	H
I	R	T	Y	Q

Cipher:

MSGIEAHRETTTTETYUIHQ

For ease in telegraphic transmission, military cipher is almost invariably broken up into five-letter groups.

MSGIE AHRET TTTET YUIHQ

Further discussions of transposition ciphers will be found in references (1, 4, 15).

In the opinion of this author, a class should spend most of its time studying the simple substitution cipher. The simple substitution cipher is written by giving each plain text letter a cipher equivalent. For example A in the plain text may become G in cipher while B in plain text becomes Q in cipher. This may be written

$$A_p = G_c$$
$$B_p = Q_c$$
$$\vdots \quad \vdots$$

The most important tools in the breaking of such ciphers are frequency counts and pattern words. E is the most common English letter and T is the second. It is hard to pick the third most common letter, but the letters of ordinary English text can be divided roughly into six groups in order of frequency.

1. E
2. T
3. A O N R I S
4. H D L F C M U
5. G Y P W B
6. V K X J Q Z

It is unusual, in messages of reasonable length, for a letter to show frequency inconsistent with its group. The actual order and frequency of individual letters may vary considerably as shown by comparing the following analysis of two distinct sets of government telegrams. Each set contains 10,000 letters.

Set 1		*Set 2*	
E—	1,367	E—	1,275
T—	936	T—	928
N—	786	R—	786
R—	760	N—	780
I—	742	O—	762
A—	738	A—	741
O—	685	I—	697
S—	658	S—	604
D—	387	D—	448
.	.	.	.
.	.	.	.
.	.	.	.
Z—	14	Z—	5

Certain patterns of letters such as the 12212 pattern of the words

"bAGGAGe," "wINNINg" and
1 2 2 1 2   1 2 2 1 2

"NOON Of"
1 2 2 1   2

are of great assistance in breaking ciphers. If word divisions are left in the message (never done in military messages) short words such as A, I, IN, OF, TO, AND, and THE, as well as common word endings such as -ING, -TION, -ERE, -IVE, and -ENT give hints of possible letter equivalents. A plausible guess at a few of these yields several letters. They can be tested by substituting in the cipher message and inconsistencies discarded.

Every time a cipher is broken, the student must collect data (frequency count for example) and make hypotheses from these data. He then uses logical reasoning to draw conclusions from these hypotheses and tests the validity of his hypotheses (and his logic) by comparing his conclusions with the data. Certainly this is excellent training either for mathematics or for life, as well as being a delightful puzzle.

As the student becomes more adept, messages with word divisions omitted will prove interesting. Extensive tables and suggestions for solving simple substitution ciphers are contained in references (4, 6, 7, 8, 9, 10, 15, 36).

Example

This cipher was found on a suspect in a jewel robbery, which took place last Saturday. [The overscore has been added to call attention to patterns within words.]

MTV NOQDGFNL TQSV UVVF
LVFM MG BGFNGF. DVVM
PL MTVKV GF DGFNQY.

In so short a message the frequency count may differ widely from normal. However we note the most frequent cipher letters are V_c, F_c, M_c and G_c. Probably E_p and T_p are represented among these. If the message concerns the robbery, some of the following words are apt to appear in the message NEW YORK, JEWEL, DIAMOND, and SATURDAY.

There is no cipher word which shows the word length and pattern of either NEW YORK or JEWEL. However the second cipher word $(\overline{\text{N}}\text{OQDGF}\overline{\text{N}}\text{L})_c$ might represent the plural form $(\overline{\text{D}}\text{IAMON}\overline{\text{D}}\text{S})_p$. Let us assume it is and see what conclusions we draw from this assumption.

Filling in the assumed letters we have

1	2	3
MTV	NOQDGFNL	TQSV
	DIAMONDS	A

4	5	6
UVVF	LVFM	MG
N	SN	O

7	8	9
BGFNGF.	DVVM	PL
ONDON.	M	S

10	11	12
MTVKV	GF	DGFNQY.
	ON	MONDA .

This looks like a valid assumption since no impossible combinations appear and the last two words (ON MONDA__)$_p$ are clearly ON MONDAY.

Of the four most frequent letters in the cipher V_c and M_c are still without equivalents. Probably one is equivalent to E_p and the other to T_p, the two most common plain text letters. If $V_c=T_p$ the fourth word is (__TTN)$_p$ and the eighth word is (MTT__)$_p$. We quickly see these are impossible words and drop the hypothesis $V_c=T_p$ and adopt its alternative $V_c=E_p$ and $M_c=T_p$ this gives

MTV	NOQDGFNL	TQSV
T E	DIAMONDS	A E
UVVF	LVFM	MG
EEN	SENT	TO
BGFNGF.	DVVM	PL
ONDON.	MEET	S
MTVKV	GF	DGFNQY.
T E E	ON	MONDAY.

The words SENT, TO and MEET appear from this, giving strength to our hypotheses. (T__E DIAMONDS)$_p$ is quite possibly THE DIAMONDS giving us $T_c=H_p$.

MTV	NOQDGFNL	TQSV
THE	DIAMONDS	HA E
UVVF	LVFM	MG
EEN	SENT	TO
BGFNGF.	DVVM	PL
ONDON.	MEET	S
MTVKV	GF	DGFNQY.
THE E	ON	MONDAY.

It is then a matter of sight reading to tell that

THE DIAMONDS HA__E __EEN SENT TO __ONDON. MEET __S THE__E ON MONDAY, is

THE DIAMONDS HAVE BEEN SENT TO LONDON. MEET US THERE ON MONDAY.

If the assumption $V_c=E_p$ (a logical one in view of its frequency count and its appearing twice as a double letter) had been made at once, the labor would have been shortened considerably.

Breaking such a cipher is like factoring a complicated polynomial. General hints can be given, but in the end intelligent guessing must be used. Each cipher requires a new attack.

A table of cipher equivalents may be carried by the parties concerned. However an easily remembered key phrase is often used to determine cipher equivalents. In this way no telltale key need be

carried. If the words NEW YORK CITY are selected as a key phrase, the phrase is written down followed by the alphabet in normal order. Repeated letters are dropped after their first appearance and the key becomes:

Plain text	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Cipher	N	E	W	Y	O	R	K	C	I	T	A	B	D	F	G	H	J	L	M	P	Q	S	U	V	X	Z

If the message in *plain text* is

MEET US AT EIGHT THIRTY

then the *cipher* becomes

DOOP QM NP OIKCP PCILPX.

The reader may be interested in reconstructing the cipher alphabet to determine the key words used in the example given in this paper.

The idea of letting the more frequent letters (E, T, A, O, etc.) have several cipher equivalents (variants) will certainly occur to the student. When he learns that Mary, Queen of Scots was beheaded in 1587 as a result of information obtained by breaking such ciphers, he may realize how insecure this innovation is.

Two time saving hints to the teacher may be in order. To encipher several messages with the same key, paste cipher equivalents on the keys of your typewriter, then hunt and peck. To encipher the same message several times in different keys, encipher it once and then continue with the alphabet in each of the columns below the enciphered message beginning each column with the enciphered letter. For example

Message	MEET	US	AT	E····
1st cipher	DOOP	QM	NP	O
2nd cipher	EPPQ	RN	OQ	P
3rd cipher	FQQR	SO	PR	Q
4th cipher	GRRS	TP	QS	R
5th cipher	HSST	UQ	RT	S
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Students deserve to be shown at least one modern cipher. The Playfair cipher will serve admirably. It was used as a field cipher during World War I. A key phrase is selected to determine a sequence of letters. We shall use the key NEW YORK CITY which was used before. The sequence is inscribed in a square divided into 25 cells. The letters I and J are considered as one letter and inscribed in the same cell.

N	E	W	Y	O
R	K	C	I-J	T
A	B	D	F	G
H	L	M	P	Q
S	U	V	X	Z

The original plain text message is broken up into pairs of letters and equivalents determined for each pair. If a pair comes out a double letter, [TT for example] it is replaced by T X T before further breaking of the message into pairs.

The cipher equivalent of each pair may be determined as follows:

Case I: If both letters appear in the same column, each letter is replaced by the letter directly below it with the convention that the bottom letter of a column is to be replaced by the letter at the top of that column. In our example KU is represented by BE.

Case II: If both letters appear in the same row, each letter is replaced by the letter directly to the right of it with the convention that the first letter of the row is the successor of the last letter in that row. In our example IR becomes TK.

Case III: If the letters appear at opposite

corners of a rectangle each letter of the pair is represented by the letter in the other corner of the rectangle in the same column with it. In our example ME becomes WL while ET becomes KO.

All possible combinations may be enciphered in this fashion.

The message MEET US AT EIGHT THIRTY becomes

Plain text	ME	ET	US	AT	EI	GH	TX	TH	IR	TY
Cipher	WL	KO	VU	RG	KY	QA	ZI	QR	TK	OI

When transmitted by telegraph (in 5 letter groups to reduce errors) this becomes

WLKOV URGKY QAZIQ RTKOI.

Note that even in this short message the five appearances of T_p are represented by O_c, G_c, Z_c, Q_c and O_c. Furthermore in repeated appearances the letters K_c, R_c, Q_c, I_c do not necessarily represent the same plain text. It may surprise the reader to learn that the Playfair cipher now has a security rating of only two or three hours in a modern properly staffed cryptanalytic office (black chamber). This does not mean that every Playfair message can be broken in this time, and certainly not that one person working alone can do it. Cryptanalysis is a time consuming occupation.

Further material on the Playfair cipher will be found in references (1, 4, 9, 10, 15, 36).

Other modern cipher methods and machines are described in references (4, 10, 14, 16, 19, 21, 22, 24, 29, 33, 36). The amazing machine called MAGIC with which the allies read Japanese cipher messages all during World War II and even after, is described in (24, 27, 30). This machine was constructed in the United States from theoretical considerations without having seen the Japanese enciphering equipment.

I urge that you give serious consideration to the possibility of teaching a unit on cryptography in *your* general mathematics course. It is an interesting topic to teach, and, especially in near war time, it is of high interest to the student—even to the student with little interest in algebra and geometry. It gives an excellent model for deductive reasoning and affords a rare opportunity to acquaint the student with statistical inference.

If you are unable to present such a unit during the present year, at least read further about ciphers yourself. If you have a mathematics club, show them the Playfair cipher and one or two others. The following descriptive bibliography contains suggestions for further reading.

The bibliography (except (2), (3) and (19)) has been selected on the basis of easy availability to the general public. Every effort has been made to cite references still in print or available in average libraries. There exist many excellent foreign works (Andre Langie's for example) all of which have been omitted. The reader may consult references (4, 5, 12, 13, 14, 16) if such bibliography is desired.

The following colleges and universities are reported to have given courses in cryptanalysis: Brooklyn, Brown, Cornell, Harvard, Hunter, Indiana, Michigan, M. I. T., Mount Holyoke, Northwestern, N. Y. U., Pennsylvania, Smith, Southern California, South Carolina, St. Johns, Syracuse, and Wesleyan. There are undoubtedly other colleges equally worthy of mention but unknown to the author. The war and navy departments offer correspondence courses in cryptography and cryptanalysis. These courses are available to members of the reserves, the regular army or navy and the national guard, and to civilian employees of the army.

Bibliography

Books

1. Ball, W. W. *Mathematical Recreations and Essays*, 11th ed. London: Macmillan, 1942.
This book, readily available in mathe-

matical libraries, contains an excellent chapter on ciphers including material on transposition ciphers as well as various substitution ciphers.

2. Friedman, William F. *Riverbank Publications* (Nos. 15 to 22).
While these papers are no longer in print, they are available at the Library of Congress and other large libraries.

3. Friedman, William F. *War Department Publications.*
These publications are not, at present, available to the general public. However, used book stores occasionally have them on sale. *Elementary Military Cryptography, Advanced Military Cryptography,* and *Military Cryptanalysis* (*Parts I and II*) are among the more interesting for the tyro. Any material by W. F. Friedman is worthy of examination.

4. Gaines, H. F. *Elementary Cryptanalysis.* Boston: American Photographic Publishing Company, 1943.
Contains many excellent suggestions on the solution of cryptograms. (The official textbook of the American Cryptogram Association.) It contains 167 assorted cipher problems, some of considerable difficulty. There is a slight tendency to lean toward the "Quintis Quigglesby's Zither" type of problem, but not sufficiently so to be objectionable. Excellent methodology.

5. Galland, J. S. *An Historical and Analytical Bibliography of the Literature of Cryptology.* Evanston, Illinois: Northwestern University Press, 1945.
Some of the material mentioned in this extensive bibliography is out of print and not easily available, but the serious student will find it comprehensive.

6. Harris, F. A. *Solving Simple Substitution Ciphers.* American Cryptogram Association (c/o W. G. Bryan, Burton, Ohio, $0.50).
This inexpensive little booklet is intended as an aid in solving problems appearing in *The Cryptogram* (see 15) and its terminology and slant are so directed. All beginners can profit by its suggestions.

7. Hitt, Parker. *Manual for the Solution of Military Cipher.* 1916. Reprinted, New York: New York University Bookstore, 1943.

8. Millikin, Donald D. *Elementary Cryptography and Cryptanalysis.* New York: New York University Bookstore, 1942.
This excellent text for the beginner has charts and tables.

9. Pratt, Fletcher. *Secret and Urgent, The Story of Codes and Ciphers.* Indianapolis: Bobbs Merrill, Inc., 1939; Garden City, N. Y.: Blue Ribbon Books ($1.00), 1942.
The publishers state that it is now "out of print" but still available in many bookstores. This book contains interesting accounts of many historically important ciphers and codes. It discusses how the economics and politics of the world have been influenced by success or failure of ciphers. It contains a wealth of background material and certainly should be read by every cryptographic tyro. The appendix contains frequency tables and notes on French, Spanish and German as well as extensive tables for the English language. It is highly recommended.

10. Wolfe, J. M. *A First Course in Cryptanalysis.* Brooklyn, N. Y.: Brooklyn College Press.
A three-volume mimeographed work covering cryptanalysis from simple substitution to Hill's algebraic transformations. This is probably the most extensive modern work on cryptanalysis available to the general public.

11. Yardley, Herbert O. *The American Black Chamber.* London: Faber and Faber, Ltd., 1931; Garden City, N. Y.: Blue Ribbon Books.
An interesting account of the title subject includes the breaking of the Japanese cipher-code during World War I and the consequent forcing of the 5:5:3 naval ratio on the Japanese.

Other References

12. "A Bibliography of Cryptography." *American Mathematical Monthly,* L (May, 1943), 345.
A short bibliography.

13. "Cipherwriting." *Encyclopedia Americana,* 1943.
Short article contains a bibliography.

14. "Codes and Ciphers." *Encyclopaedia Britannica,* 1946 edition, V, 954–59.
Descriptions of the Vigenere, Bazeries (M-94), Mechanical and other ciphers as well as examples of codes will be found in this article. A bibliography is included.

15. *Cryptogram* (c/o W. C. Bell III, 127 32nd St. N. W., Canton 9, Ohio).
A bimonthly magazine published by the American Cryptogram Association ($2.00 yearly). A few of the problems lean to the "Quintus Quigglesby's Zither" type of cipher, but articles are worthy of note.

16. "Cryptography." *Encyclopaedia Britannica,* 1946 edition, VI, 807–8.

17. Friedman, W. F. and Mendelsohn, C. J. "Notes on Code Words." *American Mathematical Monthly,* XXXIX (August–September, 1932), 394.
A good article on the composition of code (not cipher) words.

18. Green, P. D. "Lost Your Money? Wire KUBIT". *Saturday Evening Post* (November 6, 1948), 12.
This short paragraph on commercial codes is not as good as (17) and (26).

19. Gustring, B. R. "Ciphers and Ciphering Machines." *Swedish Engineers' Bulletin.* XXXVI (Oct. 1941), 6.
Although this periodical is not easily available except through large engineering libraries, it is included here as the only

unrestricted reference known to the author which contains photographs and data on Hagelin converters.

20. Helmick, L. S. "Key Woman of the T-men." *Reader's Digest*, XXXI (Sept. 1937), 51.
An account of cryptanalysis in the Treasury Department. Contains interesting background material but little on methods.

21. Hill, Lester S. "Concerning Certain Linear Transformation Apparatus of Cryptography." *American Mathematical Monthly*, XXXVIII (March, 1931), 135.
Discusses algebraic alphabets in cryptography and the Hill cipher.

22. Hill, Lester S. "Cryptography in an Algebraic Alphabet." *American Mathematical Monthly*, XXXVI (June–July, 1929), 306.
Discusses algebraic alphabets in cryptography and the Hill cipher.

23. Holstein, Otto. "The Ciphers of Porta and Vigenere." *Scientific American Monthly*, IV (October, 1921), 332.
A description and decipherment of a Vigenere cipher is the subject of this report.

24. Kobler, John. "JCHEW BISEY PYMQP UQRPD." *Collier's*, CXXVI (Oct. 28, 1950), 22–23.
Contains a description of the Vigenere cipher and a discussion of the machine called MAGIC which enabled the Allies to read Japanese ciphers during World War II. The Hagelin converters used by both sides during World War II are also described. Considerable information on the American Cryptogram Association is included.

25. Mendelsohn, Charles J. "Cardan on Cryptography." *Scripta Mathematica*, VI (Oct. 1939), 157.
Gives an interesting account of ancient cryptography, especially that of Cardan. This well-written article is both informative and entertaining.

26. "NARVO WUMND LYADI." *Nation's Business*, XXXVI (June 1948), 82.
An excellent account of commercial codes.

27. "Pearl Harbor." *Time*, XLVI (December 17, 1945), 20.
This note gives information about the machine MAGIC used during World War II to decipher Japanese messages.

28. Rhodes, J. K. "He Solves the Secrets of Cipher Writing." *American Magazine*, LXLIX (January 1925), 36–39.
An account concerning Colonel George Fabyan—cryptanalyst, and his work at the Riverbank Laboratories. The remarks concerning the Baconian cipher might be taken with a grain of salt.

29. "A Secret-code Message Machine." *Literary Digest*, LXXXIX (April 17, 1926), 22.
This short half-page description of one of the automatic tape operated polyalphabetic enciphering machines seems quite authentic.

30. *Senate Document, No. 244.*
Report of the Pearl Harbor Investigation Committee. Gives some facts on MAGIC as well as the Winds code.

31. "Uncle Sam, Cipher Wizard." *Literary Digest*, LV (November 3, 1917), 46.
This description of the decipherment of a polyalphabetic cipher in the want ads of a London newspaper affords an interesting sidelight on cryptanalysis.

32. "A War Secret." *Saturday Evening Post*, CLXLIX (October 23, 1926), 44.
An account of British divers in World War I and how they recovered certain German codes. Condensed in (34).

33. Wickware, F. S. "The Secret Language of War." *Life*, XIX (November 26, 1945), 63.
The Bazeries (M-94) and Vigenere ciphers are discussed in a short article. The four channel transmission shuffler is also mentioned.

34. "World War's Deep-Sea Spy." *Reader's Digest*, XXXI (October 1937), 74.
A condensation of (32).

35. Yardley, H. O. "Cryptograms and Their Solution." *Saturday Evening Post*, CCIV (November 21, 1931), 21.
A descriptive article containing some history and some cryptanalysis.

36. Andree, Richard V. "Cryptanalysis." *Scripta Mathematica*, XVIII (March 1952), 5–16.
This article is a discussion of cryptanalysis including frequency tables, bigrams, trigrams, and pattern words. The Vigenere Cipher is considered and Kasiski's method of breaking it is mentioned. The U. S. Signal Corp's M-94 disk cipher and the Hagelin machines are described and a simple case of Hill's excellent matrix cipher is given.

THIRTEENTH ANNUAL CHRISTMAS MEETING

National Council of Teachers of Mathematics

Hotel Lincoln, Lincoln, Nebraska

December 29-31, 1952

(Have you made your reservations? See program in the October issue.)

Number Forms: A Common Type of Synesthesia

By William E. Baldwin
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Any teacher who has from thirty to forty pupils in her classroom can reasonably expect that at least one of them will possess the type of synesthesia known as number forms. The frequency of occurrence is not definitely known, due to conflicting investigations, but it appears to be more prevalent in girls than in boys and to be somewhat associated with an imaginative approach to life's problems.

A number form is a way of perceiving the number system in space. Sometimes the form is exceedingly vivid, and the individual can describe in great detail the exact appearance and location of each digit. Sometimes the forms are in color so that the number eight, for instance, is visualized as a brilliant green. In other forms, the digits themselves are too vague to be defined, but the general background in which they are placed is definite and describable.

An example of a number form is seen in the following. The individual possessing it seems to stand at a point slightly to the right and above the first digits. They stretch out much in the same way that the ties of a railroad track appear in the foreground and disappear in the distance. In this case, the individual digits are not seen any more than the individual railroad ties are seen from a distant point of vantage. But the general areas where the digits are located can be distinguished. These areas have definite light and shade patterns. The first five digits are in a very dark area. The second five are in a bright and sunny zone. At twelve, the form curves sharply to the right in an area of gloomy darkness. At twenty, the form curves to the left into a more pleasant area, and in a wide sweep which curves back to the right approaches one hundred. Beyond one hundred the patterns of light and shade are repeated, but the form in this area takes on the vagueness of distance. It is there, but a definite effort must be made to bring it within inspection distance.

While not all number forms are this capricious, a great many of them are far more so. Logical forms, for instance, may place the digits in uniform blocks of ten, usually devoid of such extraneous features as color or light and shade. Non-logical forms may feature vast heights or abysmal depths in which the numbers advance toward a vanishing point. All number forms, however, seem to have two characteristics in common: (1) they are unchanging with time and (2) their posessors have no recollection of their beginning.

Until rather careful introspection is indulged in, the individual possessing a number form is usually not aware of its rare distinctiveness. Rather, he realizes one day that he thinks of numbers in such an organization and has been thinking of them in exactly this manner for as long as he can remember. Therefore, it can be seen that introspection of a very objective sort is necessary in order for one to ascertain whether or not he has a number form.

It can also be seen that one of the difficulties of conducting research into the phenomenon is the problem of getting people to understand what is being sought and to cause them to objectify their experiences to the extent that they can be examined. Fortunately, people who do have number forms can usually recognize their own situation when the topic is introduced, but those not in possession of them can only appear somewhat perplexed.

Due to this completely subjective nature and to the difficulty of acquiring a large enough sample to deal with, research

into number forms has been very meager. Also, potential investigators have possibly felt that a phenomenon which is experienced by such a small percentage of the population (estimates have ranged from two to five per cent) would not justify an extensive research. Finally, unless the investigator had a number form himself, enthusiasm for the subject would be difficult to arouse and sustain.

An important exception to the last point is Sir Francis Galton, the brilliant half-cousin of Charles Darwin. He investigated many forms of mental imagery, among them number forms, and published the results in his book, *Inquiries into Human Faculty*, first published in 1883 and followed by a second edition in 1907.[1] His insatiable curiosity enabled him to carry on his investigations into number forms despite the fact that he did not have one himself.

Galton devoted a considerable portion of his book to descriptions and illustrations of various number forms which had been described to him. Many of these were reproduced in color plates in his volume. His sample, which was not a representative one due to the fact that it was recruited largely from students in privileged educational institutions and members of intellectual groups, yielded the fact that the phenomenon appeared in one out of thirty adult males and in one out of every fifteen females.

In 1936, an article by Morton appeared in *The British Journal of Educational Psychology* on the number forms which appeared in a survey of 867 day-school children of Aberdeen.[2] His most important findings were that they occurred in one out of forty girls, one out of forty-seven boys, and one out of forty-three for the whole group. He also reported that the group of children who had number forms were of superior ability in arithmetic.

Two studies on number forms, both by the same investigator, were reported from Sweden. Belanner, in one article, described three cases of number forms,[3] and in the other article described the usefulness of his own number form as a mnemonic device.[4]

Studies containing the term "number forms" in their titles have been reported from Japan, but there is doubt as to whether they are concerned with number forms or merely the form of numbers.

There are several possible explanations for the existence of the phenomenon. The most obvious of these is that it is an example of learning or association. The individual simply learned numbers in a setting which he now duplicates in his number form. Such an explanation, although logical, fails completely to explain the extremely fanciful forms which are taken by some of them.

Another feature which supports the associational explanation but which could possibly be an indication of an entirely different origin is the fact that number forms appear to run in families. It can be seen that this could be supportive of the idea that they were learned, but it could also be an indication of hereditary predisposition.

Either of the above explanations, by itself, is far too superficial. A more profound and prevalent belief is that the true origin of number forms is in terms of basic perceptual organization. Werner, in discussing synesthesia in general, points out that possibly such an ability is potentially present in any mentality and is, therefore, an indication of what he calls

[1] Francis Galton, *Inquiries into Human Faculty* (2nd ed.; London: J. M. Dent and Sons, Ltd., 1907), pp. 79–105.

[2] Dan M. Morton, "Number Forms and Arithmetical Ability in Children," *The British Journal of Educational Psychology*, VI (Feb. 1936), 58–73.

[3] M. L. Reymert (abstracter), "Visuellt Talminne" (Visual Memory for Numbers), by I. Belanner, *Psychological Abstracts*, #1001, Vol. 4, (1930).

[4] M. L. Reymert (abstracter), "Hur Hjälpe Minnet för Tal Och Artal" (How to Improve Memory for Numbers and Dates), by I. Belanner, *Psychological Abstracts*, #3487, Vol. 3, (1929).

the "primitive organic unity of the senses."[5]

Whatever its cause, the phenomenon of number forms represents a comparatively unexplored area of psychological research. Its relationship to other traits of personality and intellectual ability should be investigated with the idea that it is not a freakish oddity but a further indication of the character of perceptual organization.

[5] Heinz Werner, *Comparative Psychology of Mental Development* (Rev. ed.; New York: Follett Publishing Company, 1948), p. 93.

A Note on the Use of the Slide Rule in Division

By Carl N. Shuster
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The slide rule is a very satisfactory instrument for rapid division that can be satisfied by quotients having three digit accuracy. However, there are places where six or more digit accuracy is desired. It was pointed out in a recent article in The Mathematics Teacher* that the slide rule may be used to secure five, six, eight or more digit answers to square root problems. In a somewhat similar way the slide rule may be used to secure quotients having six or more digit accuracy.

Suppose it is desired to find the quotient to $6384.27 \div 234$ accurate to 6 digits. First perform the division on the slide rule. The quotient is 27.2. Now divide on paper. It is not necessary to estimate the quotient figures, and the division is partially self checking:

```
         27.2
     ---------
234/6384.27
    1704
     662
    1947
```

(The division was done by the world method using additive subtraction to secure added efficiency and speed. Ordinary long division may be used if desired.) The quotient secured by pencil division is 27.2 and the remainder is 1947. The remainder is now divided by 234 again using the slide rule. The second quotient is 832. This annexed to the first quotient is 27.2832 which is correct to six digits.

Some of the advantages of this method are:

(1) A considerable part of the work is done on the slide rule.
(2) The results are partially self checking.
(3) The method is faster than 100% pencil computation.
(4) Where a considerable amount of division is required, it is less fatiguing than 100% pencil computation.
(5) In the cases where more than six digit accuracy is required the method may be extended to secure nine or more digits.
(6) If a 20 inch rule is used the efficiency of the method is increased and eye strain materially lessened.

* Carl N. Shuster, "Approximate Square Roots," The Mathematics Teacher, XLV (January 1952), 17–18.

An English College Course Based on the Function Concept

By E. T. Norris
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I have just recently read with great interest the article in The Mathematics Teacher entitled "What Mathematics Shall We Teach in the Fourth Year of High School?"* If my understanding of the terms is correct, it is concerned with the mathematics course immediately prior to college, and would correspond in some ways to what is taught in the sixth forms of our English grammar schools.

I am of the opinion that we in this country, by taking nearly all our sixth form pupils through a course which is essentially a preparation for an honors degree in mathematics at the university, fail to give them a sound and stimulating idea of what mathematics is and what it can do. Sir J. J. Thomson, the famous physicist, tells of one of his students at Cambridge who "had never before conceived that there was anything in mathematics that could interest any reasonable being." My own experience as a teacher in Grammar School and Teachers' Training College has convinced me that practically every student regards mathematics almost solely as a system of techniques, most of them completely isolated from each other, and that there is little or no appreciation of mathematical ideas, or even of the uses which the sciences make of mathematics. We may be quite skilled in imparting these techniques, and the student adept in acquiring them, but they should not be regarded, as too often they are, as ends in themselves. It should be possible, even at the expense of some of the techniques (although not necessarily so), to aim at a more satisfying level of achievement.

Now from time to time during the past half-century many of the most distinguished teachers of mathematics in Europe and America have stressed the importance of the idea of functionality in mathematics. There has, I believe, been a much deeper study of the implications of the function concept in mathematical education in America than in England. By and large, I think it is true to say that here relatively few mathematics teachers are aware of its importance and as a result fail to use a concept which could add significance to most of the work done in the subject at all stages of teaching.

With the greater freedom to pursue independent syllabuses which exists in our training colleges as compared with the schools, I have been able to develop during the last few years a course which makes the idea of functionality its central theme. As I know there has been a much wider development on such lines in many of the schools and colleges in the United States, I wondered whether it might be of some interest to readers of The Mathematics Teacher to know what is being attempted in one isolated instance in England. Many of the students who take the course have done mathematics only to the "School Certificate" level (an examination usually taken at the age of 15 or 16), while some have had a further one or two years preparing for the "Higher Certificate." But all students seem to come to college with rather ill-digested ideas, and with disconnected pieces of knowledge acquired from the conventional syllabuses. In the course they follow at college the development of the function concept shows the relevance of these ideas and facts, while the manipulations of algebra and calculus, for instance, are more clearly seen as the mathematician's tools rather than his finished products. The separate "subjects" in mathematics are found to be inter-related; a "tool" is taken from

* C. C. MacDuffee, "What Mathematics Shall We Teach in the Fourth Year of High School?" The Mathematics Teacher, 45 (Jan. 1952), pp. 1–5.

one field, making advances possible in quite a different direction, and this in turn opens up lines of investigation elsewhere. Throughout the study, the application of these methods to scientific thought and discovery is ostensibly paramount; but the very processes reveal the concepts of "generality" and "form" which are the essence of mathematics.

The starting point of the course is the recognition that the raw materials of the investigator are measurable quantities such as time, temperature, length, speed, and so on. These quantities, described as "variables" by the mathematician, are seen to depend on each other in different ways, and such dependences, or functional relations, may be variously expressed. The most fundamental way is by means of tabular data, but a formula may be possible, while if the relationship is between only two variables the dependence can be shown by a graph. Three important observations about these modes of representation may be made here and need to be stressed constantly during the course. First, the student is generally too much obsessed by the idea of *precise* relationships, as given in formulas. It is desirable that he should realize that numerical data is often the only possible starting point in an investigation or problem, and that the mathematical techniques devised for use with formulas are often derived from more general concepts which include both formulas and numerical data; rates of change and areas are cases in point, but there is also the general problem of finding laws to fit experimental data. Second, the formula is usually regarded solely as a means of calculation, but this is a very narrow and limited concept. The late Professor Hamley has written thus on the matter: "Let us note that the formula may be looked upon as the expression not only of a particular relation but also of a general or functional relation. The latter conception is important, but very often overlooked. The pupil should learn that, while the formula enables him to compute particular values of the 'subject' when particular values of other terms are given, it also enables him to estimate the functional significance of each term entering into the formula."[1] Third, the graph is a more understandable, if not more powerful, tool in the hands of a student than purely analytic methods, and as a "visual aid," throws light on the behavior of functions, interprets the meaning of parameters in a formula, and points to methods of analysis which otherwise tend to be abstract and incomprehensible and are merely used according to rule. Felix Klein, indeed, who was the greatest pioneer in this field of mathematical teaching, claimed that "the function concept in its graphical form should be the soul of mathematical study in the schools."[2]

In the next stage of the course an examination of well-known (or easily devised) formulas shows that there are various types of dependences, or functions, which can be expressed in general descriptive terms such as the "square law," the "inverse square law," "direct proportion," etc. One of the tasks of the course is to study in detail some of these important types. Somewhere the late Professor Nunn has a remark to the effect that no pupil should leave the study of mathematics without having some understanding of the three most important functions in practical affairs, viz., the linear, the exponential, and the periodic. In this course several others are included, e.g. the inverse proportion (or hyperbolic), and the parabolic. While the two latter may be treated in a very simple way, they contain possibilities of generalization, e.g. to the general expression of the second degree, or to the general polynomials,

[1] H. R. Hamley, "The Function Concept in School Mathematics." *The Mathematical Gazette*, XVIII (July 1934), 174.

[2] H. R. Hamley, *Relational and Functional Thinking in Mathematics*. (9th Yearbook, National Council of Teachers of Mathematics, [New York: Bureau of Publications, Teachers College, Columbia University, 1934]), p. 53.

which afford almost unlimited scope.

The standard method of dealing with any one of the simpler functions is to take a number of formulas which fall into that particular category, e.g. formulas such as $A=\pi r^2$, $s=16t^2$, exemplifying the "square law." On the analytical side it is seen that these are special examples of a general kind, which can be stated in the abstract form $y=kx^2$. On the graphical side it is found that each formula gives rise to the same type of graph, and movements of this graph in certain ways relative to the axes result in the wider generalization that the quadratic form ax^2+bx+c corresponds to a parabola whose axis is vertical, and with other positional properties depending on the value of the discriminant b^2-4ac. Quadratic equations are thus seen as special cases of the values of a quadratic function, and the extension can be made to polynomial equations in general.

Similarly the group of formulas which give rise to straight-line graphs are typical of a certain generalization. "Constants" in the abstract form of this generalization are found to have their counterpart on the graph, and this fact leads to methods of determining laws from empirical data by modification of the variables when power laws, exponential laws and others are anticipated.

The periodic functions may be introduced by examining the type of graph resulting from consideration of the displacement-time relations in vibrating springs or pendulums. The fact that a graphical form must have a corresponding analytical form leads to a definition of the trigonometric functions from a wider and more fruitful aspect than is customary. The simple trigonometric "ratios" are seen to fall within the scope of the wider definition, and afford another excellent illustration of the way mathematics advances through wider and wider generalizations which include earlier concepts as special cases. This method of approach also takes the student away from the narrow view that the trigonometric functions are necessarily associated with "angles," since the extensions to periodic motion of different kinds, usually relate to time as the independent variable. The "compounding" of two or more functions leads to a discussion of periodicity in sound, music, tidal motion, etc., and to the idea behind Fourier series. (The more limited application of the trigonometric functions to the mensurational field is an essential part of the work, of course, as well as the derivation of the important identities necessary for work involving more advanced calculus techniques.) Such a discussion of these functions, it has invariably been found, greatly enlarges the student's appreciation of what trigonometry really is, and the teacher can point to distant fields of work with other kinds of periodic functions, which reveal undreamt-of possibilities in mathematics.

This may now be an appropriate point (although not the only one) to begin to discuss more general types of functional behavior as distinct from detailed study of special kinds of dependence. The function concept, considered always from the twin aspects of formula and graph, has a natural and inevitable development in the idea of the *rate* at which a variable changes. Distance-time relations, illustrated graphically, and with emphasis on gradients of secants and tangents, serve as the starting-point for qualitative and quantitative measurements of rates of change; thus the usual techniques are evolved from actual formulas, and speeds and accelerations are understandable concepts from which the generalizations can be made. Similarly, the converse process reveals such variables as distance, speed, work, etc., as areas, and methods of calculation can then be developed. Generalization and abstraction is thus always the final stage rather than an early one. The value of both differential and integral calculus as "tools" is emphasized by their use in other fields where, without their aid, little or no progress could be made.

Discussion of various types of series is

an important part of the course (and is essential to an adequate study of the important exponential function). The functional idea is, of course, inherent in the notion of a series in which the nth term depends in some way on the value of n. A study of some of the more important types of finite series is made, including the binomial; the latter is approached, incidentally, from the point of view of polynomial laws determined from "difference" tests and does not involve a knowledge of permutations and combinations. Infinite series are then discussed, but it must be emphasized here that "rigor" is not regarded as a necessary objective; students who are not "pure" mathematicians do not, and need not, appreciate rigorous tests for convergence so long as they can see that such series are in certain cases useful and permissible and in other cases not. (Another student of whom Sir J. J. Thomson tells had ceased to attend a certain lecturer's classes because, whereas he had gone in order to learn how to use Taylor's theorem, the lecturer had talked about nothing but cases where Taylor's theorem could not be applied. The illustration, although on a different level, is apposite to much of the mathematics we try to teach to students who are not going to be "pure" mathematicians.) It should be regarded as little less than a tragedy if students do not begin to appreciate the power and usefulness of infinite series, and cases where they "work" are far more important than cases where they may not. The series which are studied are the binomial, and the direct and inverse trigonometric (the necessary techniques in calculus having been acquired), and such an enlargement of the idea of functions is a great stimulus. The uses of these series in practical computation and in the compiling of tables are dealt with, of course.

The exponential function is then "discovered," from practical instances and from simple series such as 2, 2^2, 2^3, . . . , 2^n, . . . (It is noted that there is a relation between the arithmetic series and the linear function, and an analogous relation between the geometric series and the exponential function.) It has been found that the most easily comprehended approach to "e," the natural base of the exponential function, is through the "continuous compounding" of interest on money. From the idea of a 100% rate which makes 1 unit grow into e units, there is a simple step by step development to the general formula $A = A_0 e^{kt}$, where $100k$ is the percentage rate of growth. The infinite binomial series for $Li(1+1/n)^n$ gives the series for e (the point being stressed that proof of the convergence of this series is a necessary task for the mathematician), and the same method leads to the infinite series for e^x. The essential characteristic of the exponential function, viz. that its rate of growth is always proportional to the instantaneous value of the function, can be brought out in the early stages of study and the fundamental differential equation thereby expressed. Or, through differentiation of the logarithmic function, this same property is obtained and leads to an alternative method of deriving the series for e^x by a Maclaurin expansion. There are the uses of the exponential and logarithmic series in the compilation of mathematical tables, but the main applications are to be found in the physical and biological sciences where the exponential function occurs. Cases of damped S.H.M. also illustrate how both the exponential and trigonometric functions are needed to account for certain physical phenomena.

At various points of the course it is possible to interpolate work on statistics. Such study is in line with the general theme, for fundamentally we have a statement by numerical data of the dependence between certain sets of measurements, and one of the methods employed by statisticians to investigate this dependence is graphical representation. When it is possible to express it by a normal distribution curve the mathematics of the ex-

ponential function is needed, although the actual derivation of the formula of the normal curve may not be within the scope of the work done in the course. But given the formula, the area functions associated with the curve can be calculated by integration of terms in an infinite power series. This particular section of work is especially appropriate in a training college course where an elementary study of educational statistics is a necessary part of the student's professional training.

The above is a very broad outline of the course. It is within the compass of all who have a moderate ability in mathematics, and a surprisingly large part of it is appreciated by many who would not regard themselves primarily as mathematicians. With all students a satisfying integration of the subject is achieved, and since the stress is on mathematical ideas and applications more than on techniques, interest is maintained at a high level. Although designed for the training college, and having an emphasis which enables points of contact to be made with the teaching of more elementary mathematics, it would need little adaptation in order to be suitable for the final years in the high school or as part of a general course in the first year in college.

HAVE YOU SEEN?

In *Scientific American*, August 1952
- "The Rhind Papyrus" by James R. Newman

In *Scripta Mathematica*, March 1952
- "Cryptanalysis" by Richard V. Andree
- "The Squaring of Developable Surfaces" by Michael Goldberg
- "A Hindu Approximation of Pi" by C. T. Rajagopal and T. V. Vedamurti Aiyar
- "A Survey of the Number of Copies of Newton's *Principia* in the United States, Canada, and Mexico" by Frederick E. Brasch
- "Dynamic Circles" by Herman V. Baravalle
- "Rouse Ball's Unpublished Notes on Three Fours" by H. S. M. Coxeter
- "A Sexagesimal Multiplication Table in the Arabic Alphabetical System" by Rida A. K. Irani
- "Additions to Karpinski's Trigonometry Check List" by A. W. Richeson

In *The Mathematical Gazette*, September 1952
- "25-Point Geometry" by H. Martyn Cundy
- "On the Limit of the Ratio of Sin X to X" by R. L. Goodstein
- "The Calculus Report"

In *The Australian Mathematics Teacher*, April 1951
- "Stage A in the Teaching of Geometry" by R. J. Gillings
- "Geometric Dissections" by H. Lindgren
- "The New Mathematics" by T. G. Room
- "A Suggested Approach to the Teaching of Elementary Mathematics in Junior Technical Schools" by J. H. Veness
- "Competition Rounds" by Brother Ligouri
- "The Teaching of Number-Facts" by H. Lindgren
- "The Use of Elementary Vector Methods in Trigonometry" by N. C. J. Peres

Precision—A Measure of Progress is the title of a new 63-page booklet published by the General Motors Corporation and tracing the development of measuring devices from the time of Noah to today. Another General Motors booklet, entitled *Can I be an Engineer* offers this advice to students: "Mathematics is important. Plan to take all you can in high school—the complete course, right straight through. And don't let it scare you. It's easier for some people than for others, but it doesn't take any special genius to learn all the mathematics you need. . . . Try to think of algebra or physics as something more than just a course you have to take. It's hard to do this sometimes no matter how many 'story problems' are in the book. But actually mathematics is a tool the engineer uses just as a carpenter uses his saw and hammer." Requests for these booklets should be addressed to General Motors Educational Relations Activity, Detroit 2, Michigan.

What Mathematics is Really Like was the topic of a Round Table Discussion broadcast over radio station WBIK, Chicago, on April 30, 1952 by members of the Department of Mathematics of Illinois Institute of Technology. Mimeographed copies of the broadcast will be mailed free of charge if requests are mailed before January 1, 1953 to the Department of Mathematics, Illinois Institute of Technology, Chicago, Illinois.

The House that Geometry Built

A PLANE GEOMETRY PROJECT

By NINA OLIVER
Ardmore High School, Ardmore, Oklahoma

IT WASN'T at all like a drab schoolroom anymore. There was the beauty of the late sun coming through, splashing red, blue, and yellow patterns over the floor and desks. "Seems like we ought to read a Scripture when we come in here," one of the boys in my geometry class said. This was the effect of our House That Geometry Built.

It all started back in October. We had gone along for two months without anyone coming up with a bright idea for a plane geometry project. "Mrs. Oliver, could we make designs to fit the window panes?" a girl in one of the classes asked one day. And that is exactly what we did do, working on out-of-school time from that day until late in March to complete one of the most beautiful and widely discussed projects we had ever done.

The sky was the limit, as far as design was concerned. We saw rocket ships zooming out into space, a Saturn with modernistic rings, and a Stairway to the Stars take form on our exhibit board. Most designers were rather conventional, however, staying in the bounds of reality.

We swamped the five and ten cent stores, stationery shops, and drug stores, for cellophane paper and other necessary materials, until finally salespeople knew we weren't crazy when we ordered a dozen rolls of cellophane wrapping paper in every color available.

The plan to make transparencies for the windows was in its inception very hazy, but it caught on in every section of the class. The question of measuring the windows and reducing the dimensions to the size of notebook paper was an assignment which required climbing a ladder and taking careful readings on rulers and tape lines. Every student got into the game of solving these similar figures by ratio and proportion without knowing that it had such a scientific name.

There are nine windows in the room, six large, very tall, sunny ones on the south, and three shorter ones above the blackboard on the west. Since the dimensions of the two kinds of windows were different, it was necessary to know which window each student was designing. This brought up the plan of dividing the group into nine committees, the chairman of each elected by his class and the members chosen by the chairman. Each class had three committees with about ten members in each. Since the work was all to be done on marginal time, the committees had fun meeting in the home of the chairman for evenings of cooperative work. These meetings were held frequently throughout the whole project, because all of the planning and most of the actual construction were done on off-school time by the students acting in committees.

The last day of the first semester was the deadline set for submitting designs. Everyone waited anxiously for exhibit time to begin. No one seemed to be contented with his first effort. Consequently, by the time the deadline arrived we had over 150 worthy designs.

This assortment of creations was exhibited, criticized, and compared for a week. Then it became the difficult task of each committee to eliminate all but two of its designs. The student whose creation was selected felt highly honored, since only two designs were used by a group in developing a whole window. It took no special knowledge of art on the part of the teacher or students to conceive and develop this project. In fact, some of the winning designs were made by students

Closeup of three of the large windows. Dimensions are $9' \times 3'$.

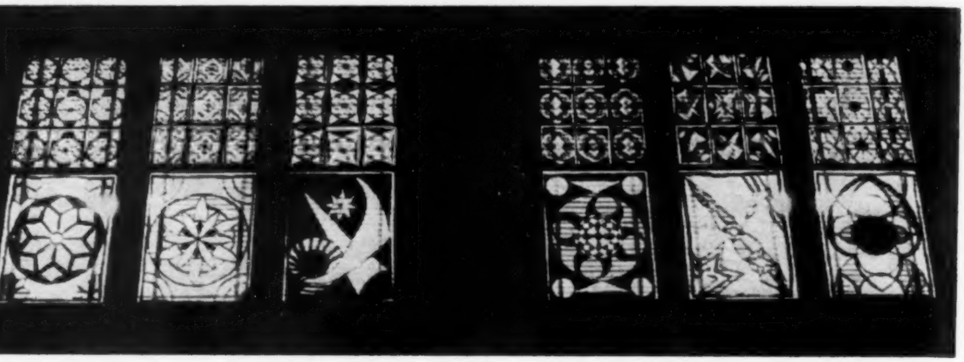
who had never studied art.

The problem of selecting the basic material from which to cut the designs was a knotty one. The material had to be durable and tough and as translucent as possible, and had to come within our budget. Several kinds of paper were tried out with different kinds of glue and paste, and at last it was decided to use a tough, light brown paper bought from a lumber yard. The glue and paste both proved to be disappointing because they made the cellophane pucker, so in the final stages of construction, Scotch tape was used to attach the cellophane to the construction paper.

Each student brought ten cents to put into an expense fund, because we learned that one student's dime would buy very little. A buyer in each class drew upon this money when materials were needed. We investigated the cellophane market early in order to be sure of an ample supply. Blue proved to be the favorite color, and when a critical shortage developed, an order for a large quantity was rushed to Oklahoma City.

The actual construction of the transparencies was a series of distinct jobs. First, the heavy paper had to be squared up and cut out; then the design must be enlarged and drawn upon it. This process involving many principles of geometry, for example, the quickest way to find the center of a large sheet of paper is the fact that the diagonals of a parallelogram bisect each other. The longest and most tedious part of the whole project was that of cutting the designs out of the heavy paper with razor blades. Every transparency was made double to make it durable and as pretty from the outside as on the inside. There were twenty designs for each committee to cut out. The top half of each window is composed of nine small panes, and the bottom half, of one large one. One of the designs selected was used for the nine small panes and the other for the large one. As can be seen in the pictures, some unexpected effects were produced by the repetition of the patterns in the arrangement of nine panes.

The members of one committee pledged



Time exposure taken at night from street level.

themselves to be first in mounting their transparency on the window. They stayed after school, cutting and pasting until six o'clock, knowing what a big surprise their window would give the classes next day. When they finally climbed the ladder and fastened the lovely pattern to the window they were thrilled with the joy of a dream coming true.

The unveiling of this first window had the desired effect. It was like keeping up with the Joneses; everyone had to have his window up as fast as his hands could move. So before the week was over we sat in a lovely retreat surrrounded by the fruits of our labor and imagination. Everyone felt proud and contented, strutting with a feeling of accomplishment.

We agreed that late afternoon was the ideal time to view the windows, because the room faces the southwest. There were groups who came to see the effect of the sun on this array of bright colors in cellophane every afternoon. It was natural for viewers to state their preference and to be able to explain why they liked certain windows, so we decided to have an Open House and give our admiring public a chance to express its collective opinion.

March 17 was the day decided upon. Personal invitations were issued to adults to come between 4:00 and 5:30 P.M.

Planning for the Open House was like launching another project, but the committees swung into action and we soon had things moving along. They baked enough cookies for 700 guests, prepared ballots for everyone in town, and sent out invitations to the members of the art clubs and to the architects. They dressed a big table in a lace cover, provided colorful napkins, and brought big trays for their cookies. When the guests began to arrive, one group of students served refreshments, another gave each guest a ballot, and still another showed them around the room. Guests were encouraged to read the legend under each window. This legend bore the name of the designer and that of the committee members who made the transparency.

"Whether you vote for Taft or Kefauver,
Look these windows carefully over,
And vote for the one which according to you,
Will win the election of '52."

This very broad-minded invitation to vote was handed to each guest as he entered the room. He deposited it in the regulation county ballot box as he left. Over 400 voters filed through the room before school, at noon, and from 4 until 5:30 that day. Tabulation of the votes to determine the first, second, and third place winners, was made the next day. We should have prolonged the election, however, as later all of the mathematics and art classes from our junior high school came in to study the windows as a part of their school work. These classes held their own poll and sent us their decision.

We have had some very distinguished visitors in our House That Geometry Built. We have been honored by having

the local Classroom Teacher Association hold the social hour of their meeting in the "House." Artists from over the whole state, in Ardmore for the Lake Murray Art Festival, came to see the windows.

The fact that geometry is moving in such elegant circles surprises some people, but not us who know its possibilities. We know of no one who would not profit from a year spent studying geometry, whether he be architect, artist, lawyer, preacher, salesman, bricklayer or housewife. Few arguments have to be made for it, so apparent is its value. First of all, it lies at the basis of the whole science of measurement. Then too no study can quite take the place of geometry as a cure for slovenly habits of thinking.

More parents and patrons visited our school during this exhibit than have visited us in years. When ninety proud youngsters go out into the community telling their relatives and friends to visit a specific school room for a specific purpose, they will come, and they will go away "satisfied customers." Some of the mothers requested that the lights be turned on at night so that their husbands and friends, who could not visit during the day, might see the windows from the outside. The arrangement was made for one night, and the time-exposure appearing in this text shows what they saw from the street level. This request is an added indication that a project of this type is good public relations, as well as an opportunity for group planning.

Visiting a Computation Laboratory

By Fred Gruenberger

Computing Service, University of Wisconsin, Madison, Wisconsin

Numerous articles are appearing about automatic high speed computing machines. The art of machine calculation is still in its infancy, but is having a tremendous effect on the field of mathematics, both pure and applied. The first large-scale high speed electronic calculator (ENIAC) was put into operation in 1946. At the present time there are over 40 giant calculators (EDVAC, SEAC, SWAC, UNIVAC, etc.) now in operation in this country; new ones are put into operation at the rate of about one a month. Not all of them are electronic. Smaller electronic machines (Card Programmed Calculators, or CPC for short) made by the International Business Machines Corporation (IBM) are in wide use; over 150 of them are in operation. Indeed, any industry whose payroll numbers several hundred persons probably has an automatic punched card calculator.

The University of Wisconsin has in its Computing Service both a Card Programmed Calculator and a small electromechanical calculator, both of which are punched card machines. During the University's spring recess, the four Madison high schools are invited to bring a group of senior mathematics students to see the equipment in action.

There is probably an automatic calculator at work in any town of over 25,000 population. The local office of IBM can arrange for your students to see the machines in action.

The whole topic of modern high speed computing equipment is one of those where some of the students may know a great deal more than their teacher. Whether or not this is the case, it might be well for the teacher to arrange to see the equipment in advance, if only to prepare the students for what they are to see. Reference books such as Berkeley's *Giant Brains*, or *High Speed Computing Devices* by the staff of Engineering Research Associates, Inc., will be helpful.

APPLICATIONS

Edited By SHELDON S. MYERS
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AS IMPORTANT as general education may be it has always been my concern that, through over-emphasis on the general and the average, we may tend to neglect the gifted students who will provide the basis of our nation's technical supremacy in the future. Therefore, from time to time it may be appropriate in this department to include applications of a nature that will stimulate and challenge this type of student. Below are found two applications of one of the most fundamental patterns of mathematical reasoning, the principle of mathematical induction, which may profitably be brought to the attention of gifted juniors and seniors.[1]

C. Al. 6 Gr. 10-14 *An Application of Mathematical Induction to the Tower of Hanoi Puzzle*

In the October issue of THE MATHEMATICS TEACHER, page 505, there was described an experiment whereby the formula relating the number of discs to the number of moves in the Tower of Hanoi Puzzle was derived empirically by induction. We shall now prove this formula, $M_n = 2^n - 1$ (where M_n is the number of moves for n discs), by mathematical induction. We also hope to show in the process that mathematical induction is a purely demonstrative method and has nothing to do with experimental induction. An outline of the following proof was submitted by MILTON GLANZ, a graduate student at Ohio State University.

Rules of the Tower of Hanoi Puzzle used as postulates in the proof:

1. Object: Find the minimum number of moves for transferring all the discs from post 1 to post 2 or 3 so that all the discs are on a new post in the same order as on post 1.
2. Each move consists in moving one and only one disc from one post to another.
3. No disc can be placed on top of a smaller one.

It can easily be shown that a minimum solution must exist, since the set of all solutions for any number of d discs represents a non-empty set of positive integers and a fundamental property of such a set is that it must contain a smallest member.

Now suppose there are d discs on post 1. In order to transfer the d discs to post 2, it is necessary that the discs above the bottom one, or $d-1$ discs be transferred to post 3, so that the largest disc on the bottom of post 1 may be moved to the empty post 2 (rule 3).

Let us assume that k is the minimum number of legal moves needed to transfer the $d-1$ discs to post 3. Note that k is dependent only on the $d-1$ discs, not on the largest disc. After the largest disc has been moved to post 2, we have used $k+1$ moves. For a solution it is now a necessary and sufficient condition that the $d-1$ discs on post 3 be transferred on top of the largest disc on post 2. This requires another k moves, or $2k+1$ altogether.

Thus we have shown two things: (a)

[1] Three reliable sources which discuss reasoning by mathematical induction are R. Courant and H. Robbins, *What is Mathematics?* (New York: Oxford University Press, 1941), pp. 9–20; J. W. Young, *Lectures on Fundamental Concepts of Algebra and Geometry* (New York: Macmillan, 1911), pp. 67–78; M. R. Cohen and E. Nagel, *An Introduction to Logic and Scientific Method* (New York: Harcourt, Brace, 1934), pp. 147–148.

If there is a solution for $d-1$ discs, there is also a solution for d discs; and (b) If the minimum solution is k moves for $d-1$ discs, then the minimum solution for d discs is $2k+1$ moves.

We can now show that the puzzle has a solution for every integral number of blocks. For, if $d-1=1$, that is when there is only one disc on post 1, k is obviously 1. By (a) above, if there is a solution for 1 disc, there must be one for 2 discs, and so on for every integral number of blocks. This is an illustration of mathematical induction within the main proof. We are now ready for the main proof.

Mathematical induction in this case requires two things to be done: (c) If the theorem, $M_n=2^n-1$, is assumed true for $n=r$, then it is shown to be true for $n=r+1$, and (d) Show the theorem is true for $n=1$.

(1) Assume $M_r=2^r-1$ is true [(c) above].

In (b) above it was shown that if k are the moves for $d-1$ discs, $2k+1$ are the moves for d discs. In like manner if M_r are the number of moves for r discs then:

(2) $$M_{r+1}=2M_r+1.$$

Substituting (1) in (2)

$$M_{r+1}=2(2^r-1)+1$$

or

(3) $$M_{r+1}=2^{r+1}-1.$$

The result in (3) shows that the form in (1), which was assumed to be true, continues to be true when extended to the next member of the series, $n=r+1$.

Finally when $r=1$, it is obvious that $M=1$, and these values satisfy the original theorem. If it is true for $n=1$, it must be true for $n=2$ by result (3), and so on for all subsequent values of n. It has now been established as generally true, whereas the experimental derivation in the November 1951 issue could be absolutely depended upon only for the first five discs.

C. Al. 7 Gr. 10–12 *Installment Loans*

In 1941, Mr. A sold Mrs. B a house and lot for \$4500, \$700 down and the balance in equal monthly installments of \$32, each installment to be applied first to pay simple interest at 6% on the unpaid balance and second to reduce the balance. Thus at the end of the first month, \$19 of the monthly installment went for interest and \$13 went to reduce the unpaid balance from \$3800 to \$3787. Mrs. B made the payments regularly at the end of each month for ten years. Then she asked Mr. A what her unpaid balance was so she could pay off the loan. Mr. A had kept no records, except the fact that all installments had been paid. He took the problem to his local banker who told him that he would have to apply each of the 120 installments in turn in order to discover the amount of unpaid balance. At least this banker knew of no other way.

Nevertheless the algebra required to solve the problem is well within the ability of a good high school student in advanced algebra.

Let B_0 be the original unpaid balance.
B_r be the unpaid balance after r installments.
I be the amount of each installment.
i be the interest rate per installment period.

At the end of the first installment period the principal is B_0, the interest is iB_0, the amount applied to the unpaid balance is $I-iB_0$, and the unpaid balance is

$$B_1=B_0-(I-iB_0)$$
$$=B_0+iB_0-I.$$
$$B_1=B_0(1+i)-I.$$

At the end of the second installment period the principal is B_1, the interest is iB_1, the amount applied to the unpaid balance is $I-iB_1$, and the unpaid balance is

$$B_2=B_1(1+i)-I.$$

Substituting the expression above for B_1

$$B_2 = B_0(1-i) - I[1+(1+i)].$$

At the end of the rth installment period the unpaid balance is

$$B_r = B_0(1+i)^r - I[1+(1+i)+(1+i)^2 + \cdots + (1+i)^{r-1}].$$

This formula can readily be proved through mathematical induction by first obtaining the formula for B_{r+1} directly and comparing it with the formula for B_r with $r+1$ substituted for r. B_{r+1} is obtained directly as follows:

$$B_{r+1} = B_r(1+i) - I.$$

The expression for B_r is now susbtituted for B_r in the above equation.

$$B_{r+1} = B_0(1+i)^{r+1} - I[1+(1+i)+(1+i)^2 + (1+i)^3 \cdots (1+i)^r].$$

This expression agrees with the one obtained by substituting $r+1$ for r in the expression for B_r.

In addition it must be shown that B_r is true for the case $r=1$. When 1 is substituted for r in the expression for B_r, the following expression is obtained:

$$B_1 = B_0(1+i) - I.$$

Since this agrees with the expression for B_1 which was derived by other means, the expression for B_r has now been established by mathematical induction.

The quantity in brackets is a geometric progression of r terms with the first term 1 and common ratio of $(1+i)$ Its sum is

$$S_r = \frac{(1+i)^r - 1}{i}.$$

Therefore,

$$B_r = B_0(1+i)^r - I\,\frac{(1+i)^r - 1}{i}.$$

Or,

$$B_r = B_0 A_r - I S_r$$

where A_r is the amount of $1 at compound interest at rate i per interest period for r periods, and S_r is the amount of an annuity of $1 at the end of each interest period for r periods at the rate i.

If the student is familiar with interest tables, A_r and S_r may be found directly in the tables. If not, the formula may be put in the form

$$B_r = \frac{I - (I - iB_0)(1+i)^r}{i}.$$

The value of $(1+i)^r$ may now be computed by logarithm tables.

If one wishes to find the number of installments needed to retire a loan B_0 at rate i per installment period, one need only set $B_r = 0$ in the above expression.

$$(1+i)^r = I/(I - iB_0).$$

$$r = \frac{\log\,[I/(I - iB_0)]}{\log\,(1+i)}.$$

Another question of importance to income-tax-paying home owners is the following. How much interest is paid during any given number of installment periods of the life of a loan? To be specific, how much interest is paid in the first s installments following the rth installment?

The interest for the $(r+1)$th period is

$$iB_r = I - (I - iB_0)(1+i)^r$$

and for the $(r+2)$th period is

$$iB_{r+1} = I - (I - iB_0)(1+i)^{r+1}$$

and for the $(r+s)$th period is

$$iB_{r+s-1} = I - (I - iB_0)(1+i)^{r+s-1}.$$

Hence the interest for the first s periods after the rth period is

$$I_{s-r} = sI - (I - iB_0)\,\frac{(1+i)^{r+s} - (1+i)^{r+s-1}}{i}$$

$$= sI - (I - iB_0)(1+i)^r\,\frac{(1+i)^s - 1}{i}.$$

$$I_{s-r} = sI - (I - iB_0)A_r S_s.$$

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REFERENCES FOR MATHEMATICS TEACHERS

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Calculus in the Classroom

1. NATURE OF THE CALCULUS

Abbott, P. *Teach Yourself Calculus.* London, The English Universities Press, Ltd., 1946; 380 p.

Bell, E. T. *Mathematics: Queen and Servant of Science.* New York, McGraw-Hill, 1951; 437 p. Chapt. 15: pp. 320–41.

Boyer, Carl. *The Concepts of the Calculus.* New York, Columbia University Press, 1939; 346 p.

Boyer, Carl. "History of the Derivative and Integral of the Sine." MATHEMATICS TEACHER 1947, 40: 267–75.

Brown, F. G. *Everyman's Mathematics.* London, Angus & Robertson, 1947; 747 p. Calculus: pp. 629–80.

De Morgan, Augustus. *Elementary Illustrations of the Differential and Integral Calculus.* Chicago, Open Court Publishing Co., 1899; 144 p.

Fehr, Howard. *Secondary Mathematics: A Functional Approach for Teachers.* Boston, D. C. Heath, 1951; 431 p. The derivative: pp. 168–90; integration: pp. 315–37.

Hogben, Lancelot. *Mathematics for the Million.* New York, W. W. Norton, 1937; 647 p. "What the Calculus is About": pp. 507–70.

Hooper, A. *The River Mathematics.* New York, H. Holt, 1945; pp. 340–79.

James, Glenn. "The Gist of the Calculus." *Mathematics Magazine,* 1948, 22: 29–46.

Kasner, E. and Newman, J. *Mathematics and the Imagination.* New York, Simon and Schuster, 1940; 380 p. "Change and Changeability—the Calculus": pp. 299–356.

Klein, Felix. *Elementary Mathematics from an Advanced Standpoint: Arithmetic; Algebra; Analysis.* (Trans. by E. R. Hedrick and C. A. Noble.) New York, Macmillan, 1932; 274 p. Infinitesimal calculus: pp. 207–36.

Kramer, Edna. *The Main Stream of Mathematics.* New York, Oxford University Press, 1951; 321 p. "A world in flux": pp. 209–40.

Mercer, J. W. *The Calculus for Beginners.* Cambridge University Press, 1920; 440 p.

Miller, Denning. *Popular Mathematics.* New York, W. H. Wise & Co., 1942; 616 p. "The brightest pebble on the beach": pp. 513–85.

Moore, J. H. and Mira, J. A. *The Gist of Mathematics.* New York, Prentice-Hall, 1942; 726 p. "Limits": pp. 560–79.

Richardson, M. *Fundamentals of Mathematics.* New York, Macmillan, 1941; 525 p. "Limits and the calculus": pp. 288–333.

Sawyer, W. W. *Mathematician's Delight.* New York, Penguin Books, 1946; 215 p. The calculus: pp. 107–60.

Swenson, John. *Integrated Mathematics with Special Application to Analysis.* Ann Arbor, Mich., Edwards Bros., 1935; 473 p.

Thompson, S. P. *Calculus Made Easy.* New York, Macmillan, 1910.

Titchmarsh, E. C. *Mathematics for the General Reader.* London and New York, Hutchinson's University Library, 1950; 160 p. Calculus: pp. 124–55.

Van Horn, C. E. *A Preface to Mathematics.* Boston, Chapman and Grimes, 1938; 124 p. The calculus: pp. 76–116.

Whitehead, A. N. *An Introduction to Mathematics.* New York, H. Holt, 1911; 252 p. The differential calculus: pp. 217–35.

2. CALCULUS IN THE HIGH SCHOOL

Farmer, Susie B. "The Place and Teaching of Calculus in Secondary Schools." MATHEMATICS TEACHER, 1927, 20: 181–202.

Fehr, Howard. "The Value of Analytics and Calculus in the Secondary School." MATHEMATICS TEACHER, 1934, 27: 296–302.

Goff, Robert R. "A Few Lessons in Calculus for High Schools." MATHEMATICS TEACHER, 1922, 15: 307–8.

Kinney, J. M. "Calculus in the High School." MATHEMATICS TEACHER, 1923, 16: 321–31.

Knowles, W. "Teaching of Easy Calculus to Boys." *Mathematical Gazette,* 1913, 7: 201; 1914, 7: 322.

MacDuffee, C. C. "What Mathematics Shall We Teach in the Fourth Year of High School?" MATHEMATICS TEACHER, 1952, 45: 1–5+.

Mathematical Association (England). *The Teaching of Calculus in Schools.* London, G. Bell and Sons, 1951; 76 p.

Nordgaard, M. "Earlier Place for the Calculus in the Curriculum." MATHEMATICS TEACHER, 1927, 20: 321–27.

Nordgaard, M. "Introductory Calculus as a High School Subject." *National Council of Teachers of Mathematics,* Third Yearbook, 1928, pp. 65–101.

Rosenberger, Noah B. "An Outline of High

School Calculus." *School Science and Mathematics*, 1930, 30: 937–44.

Rosenberger, Noah B. *The Place of the Elementary Calculus in the Senior High School Mathematics.* New York, Teachers College, Columbia University, 1921; 81 p.

Rosenberger, Noah B. "Place of Elementary Calculus in Senior High School Mathematics." MATHEMATICS TEACHER, 1922, 15: 152–56.

Swenson, John. "Selected Topics in Calculus for the High School." *National Council of Teachers of Mathematics*, Third Yearbook, 1928, pp. 102–34.

Tyler, W. H. "Calculus for Schools." MATHEMATICS TEACHER, 1922, 15: 208–11.

3. Organization and Content

Ballard, Ruth. "Teaching of Mathematics at the Junior College Level." *School Science and Mathematics*, 1941, 41: 482–86.

Boeker, M. D. *Status of the Beginning Calculus Students in Precalculus College Mathematics.* New York, Teachers College, Columbia University, 1947; 83 p.

Boyer, Carl. "The First Calculus Textbooks." MATHEMATICS TEACHER, 1946, 39: 159–67.

Crosland, L. "Introduction of Numerical Calculus." *Journal of Education* (London), 1932, 64: 598–99.

Fagerstrom, W. H. *Mathematical Facts and Processes Prerequisite to the Study of the Calculus.* New York, Teachers College, Columbia University, 1933; 68 p.

Johnson, C. A. "Coordinating Calculus Instruction in the Engineering Program." *Journal of Engineering Education*, 1951; 41: 537–41.

MacDuffee, C. C. "Objectives in Calculus." *American Mathematical Monthly*, 1947, 54: 335–37.

Osgood, W. F. "The Calculus in Our Colleges and Technical Schools." *Bulletin, American Mathematical Society*, 1907, 13: 449–67.

Parker, James E. "The Teaching Objectives in a First Course in the Calculus." MATHEMATICS TEACHER, 1944, 37: 347–49.

Scheffé, H. "At What Level of Rigor Should Advanced Calculus for Undergraduates be Taught?" *American Mathematical Monthly*, 1940, 47: 635–40.

Simons, Lao G. "The Adoption of the Method of Fluxions in American Schools." *Scripta Mathematica*, 1936, 4: 207–19.

Society for the Promotion of Engineering Education. *Syllabus of Mathematics; A Symposium.* Ithaca, New York, The Society, 1912; 136 p. A syllabus of differential and integral calculus: pp. 75–118.

Woods, Roscoe. "How Can Interest in Calculus Be Increased?" *American Mathematical Monthly*, 1929, 36: 28–32.

4. General Teaching Techniques

Alfred, Brother Bernard. "Another Start." *National Mathematics Magazine*, 1938–39, 13: 133–36.

Butler, C. H. and Wren, F. L. *The Teaching of Secondary Mathematics.* New York, McGraw-Hill, 1951; pp. 508–40.

Couillard, B. "Mechnical Processes in Introductory Calculus and Analytic Geometry." *School Science and Mathematics*, 1926, 26: 921–30.

Dockeroy, N. R. C. "The Teaching of Mathematical Analysis." *Mathematical Gazette*, Dec., 1935.

Douglas, J. A. "Graphical Analysis Used to Teach Motion." *School Science and Mathematics*, 1938, 38: 135–40.

Dragoo, R. C. "Teaching the Calculus." *National Mathematics Magazine*, 1945, 19: 186–93.

Goodstein, R. L. "Teaching the Calculus." *Mathematical Gazette*, Oct., 1941.

Hatch, Hollis. "Calculus Looks at the Barometer." *National Mathematics Magazine*, 1943, 18: 127–29.

Humbert, R. "Wanted: Better Mathematics Instruction." *School Science and Mathematics*, 1948, 48: 534–35.

Huntington, E. V. "Teaching the Calculus." (Four Papers on the Teaching of Mathematics.) *Society for the Promotion of Engineering Education*, Bulletin 19, April, 1932; pp. 35–70.

Huntington, E. V. "Teaching the Calculus." *Journal of Engineering Education*, 1932, 22: 485–521; 708–10.

Lange, Louise. "More Stress on General Formulations in Calculus Problems." *American Mathematical Monthly*, 1950, 57: 181–83.

Myers, Sheldon S. "The Meaning of Derivative and Integral Illustrated by the Speedometer." MATHEMATICS TEACHER, 1952, 45: 58–59.

Nunn, T. P. *The Teaching of Algebra (Including Trigonometry).* New York, Longmans, Green & Co., 1919; 616 p. Differentiation and integration: pp. 541–66.

Olds, Edwin G. "We Discover the Meaning of Curvature." *National Mathematics Magazine*, 1940, 14: 469–71.

Olds, Edwin G. "A Fresh Start." *National Mathematics Magazine*, 1937–38, 12: 290–93.

Phillips, E. G. "Differentiation from First Principles." *Mathematical Gazette*, 1938, 22: 374–76.

Ransom, W. R. "A Geometrical Meaning for $f''(x)$." *School Science and Mathematics*, 1943, 43: 600.

Turnbull, C. "Logical Explanation of the Calculus." *Electrician*, July 13, 1945, 135: 34–35.

Wren, F. L. and Garrett, J. "Development of the Fundamental Concepts of the Infinitesimal Calculus." *American Mathematical Monthly*, 1933, 40: 269–81.

5. The Limit Concept

"First Encounter with a Limit." *Mathematical Gazette*, 1935, 19: 109–23, 131–34.

Gibbins, N. "Chapter on Limits." *Mathematical Gazette*, 1935, 19: 249–55.

Gordon, David. "An Historical Introduction to the Theory of Limits." *Bulletin, Association of Teachers of Mathematics of New York City*, Jan., 1945, 1: 3–7.

Jackson, D. "Notion of Limit." MATHEMATICS TEACHER, 1924, 17: 72–77.

Jackson, D. "Variables and Limits." MATHEMATICS TEACHER, 1916, 9: 11–16.

Nunn, T. P. "The Arithmetic of Infinites." *Mathematical Gazette*, 1910–11, 5: 345–56, 377–86.

Roe, E. D. "A Generalized Definition of Limit." MATHEMATICS TEACHER, 1910, 3: 43–49.

"Some Notes on the Limit Concept." *Mathematics Magazine*, 1952, 25: 163–65.

6. DIFFERENTIALS

Barton, A. "Differentials from a New Viewpoint." *Mathematical Gazette*, 1945, 29: 193–99.

Combridge, J. T. "The Teaching of Differentials." *Mathematical Gazette*, 1932, 16: 7–8.

Hadamard, J. "Notion de différentielle dans l'enseignement." *Mathematical Gazette*, 1935, 19: 341–42.

Longley, W. "First Exercises on Differentials." *National Mathematics Magazine*, 1936, 10: 219–26.

Phillips, E. G. "The Teaching of Differentials." *Mathematical Gazette*, 1931, 15: 401–03.

Picken, D. "On Differentials." *Mathematical Gazette*, 1935, 19: 79–86.

Temple, E. "The Rehabilitation of Differentials." *Mathematical Gazette*, 1936, 20: 120–31.

7. INTEGRATION METHODS

Folley, K. W. "Integration by Parts." *American Mathematical Monthly*, 1947, 54: 542.

Ingersoll, B. M. "Geometric Derivation of the Formula for Integration by Parts." *National Mathematics Magazine*, 1944, 18: 280–83.

Jerbert, A. R. "The Art of Summation." *School Science and Mathematics*, 1946, 46: 703–7.

MacNeish, H. F. "Logarithmic Integration." *American Mathematical Monthly*, 1949, 56: 25–27.

Olds, C. D. "Remarks on Integration by Parts." *American Mathematical Monthly*, 1949, 56: 29–30.

Reynolds, J. B. "Undetermined Coefficients in Integration." *American Mathematical Monthly*, 1947, 54: 37–38.

Smith, H. W. "Some Integral Formulas." *American Mathematical Monthly*, 1949, 56: 27–28.

Whitman, E. A. "The Film: A Triple Integral." *American Mathematical Monthly*, 1942, 49: 399–400.

Whitman, E. A. "The Use of Models When Teaching Triple Integration." *American Mathematical Monthly*, 1940, 47: 45–48.

8. SPECIAL TOPICS

Adler, Claire. "Calculus *versus* Geometry." MATHEMATICS TEACHER, 1938, 31: 19–23.

Burton, W. W. and Miller, W. G. "The Integral of Powers of the Secant and Cosecant." *American Mathematical Monthly*, 1941, 48: 544–45.

Laws, L. S. "A Simplification of the Second Derivative Test." *American Mathematical Monthly*, 1947, 54: 543.

Lubin, C. I. "Differentiation of the Trigonometric Function." *American Mathematical Monthly*, 1947, 54: 465–67.

MacDonald, J. K. L. "Elementary Rigorous Treatment of the Exponential Limit." *American Mathematical Monthly*, 1940, 47: 157–59.

Mancill, J. D. "One Sided Maxima and Minima." *American Mathematical Monthly*, 1948, 55: 311–15.

Menger, Karl. "Methods of Presenting e and π." *American Mathematical Monthly*, 1945, 52: 28–29.

Menger, Karl. "On the Teaching of Differential Equations." *American Mathematical Monthly*, 1944, 51: 392–95.

Olds, Edwin G. "Teaching the Concept of Directional Derivative." *National Mathematics Magazine*, 1936, 10: 268–75.

Ransom, W. R. "Introducing $e=2.718+$." *American Mathematical Monthly*, 1948, 55: 572.

Rickey, F. A. "A Problem in Minimum Values." *National Mathematics Magazine*, 1939, 13: 362–66.

Robson, A. "Sign Conventions and Curvature." *Mathematical Gazette*, 1946, 30: 1–5.

Scheffé, H. "Series Comparison Test for Calculus Students." *American Mathematical Monthly*, 1941, 48: 256–57.

Turton, F. J. "A Lesson on π and on $\lim_{\theta\to 0}$ $(\sin \theta/\theta)$." *Mathematical Gazette*, 1946, 30: 282–86.

Walsh, J. L. "A Rigorous Treatment of the First Maximum Problem in the Calculus." *American Mathematical Monthly*, 1947, 54: 35.

Correction: On page 236 of the article "A New Approach to the Trisectrix of Maclaurin" by Juan E. Sornito in the March 1952 issue, change $DE=EO=EF=r$ to $DE=EO=OF=r$.

Two bibliographies on mathematics education prepared by Dr. Kenneth E. Brown are as follows: "Selected Bibliography of Current Articles in Mathematics Education" and "Selected Bibliography of Reference and Enrichment Material for the Teaching of Mathematics." They are listed as Circular Nos. 346 and 347 respectively. Copies may be obtained by writing to Dr. Brown, Specialist for Mathematics, Federal Security Agency, Office of Education, Washington 25, D. C.

MATHEMATICAL MISCELLANEA

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60. *Big Numbers—An Answered Letter*

In *Miscellanea 50* [THE MATHEMATICS TEACHER, XLV (April 1952), p. 270] we published an inquiry by Tom Hawk about the highest actual number known to man. Here are some answers.

135-12 77th Avenue
FLUSHING 67, N. Y.
April 14, 1952

Mr. Tom Hawk
1752 Long Valley Road
Glenview, Illinois

DEAR TOM:

After reading your letter in THE MATHEMATICS TEACHER, I decided to answer your query. You have asked a question which I asked my teachers years and years ago. Unfortunately, my teachers were impatient with me, and I had to dig very hard for an answer. And, what do you know? I still cannot find an answer to the question "What is the highest actual number known to man?"

You see, Tom, "number" is an invention of Man. We invented "number" long ago (this invention was never patented, so everybody can use it, and we make good use of it).

It may seem somewhat strange to you, Tom, but your question is one of the most important questions in mathematics. Many a mathematician asked the same question and could not obtain the correct answer. But, in the processes of trying to find a satisfactory answer, mathematicians made some very important discoveries. So, do not get discouraged. You have within you the seeds of good and sound mathematical thinking. If someone tells you that your question is silly, laugh at them. Some questions are not as silly as they may sound. Go on asking all the questions that may pop up. This is the way progress is made.

"Number," Tom, is an invention which enables us to count and tell "how big," "how small," and "how many" objects there are in a collection of objects. Those collections which contain *the same amount of objects* are assigned the same number. Thus, the collection of the fingers on one hand, the collection of the petals of an apple blossom, the collection of the letters in the word TABLE, all of such collections are assigned the number *Five*.

If we want to obtain a larger collection of objects, we add one more object to the collection we have. Thus, if we add the letter S to the word TABLE we obtain the word TABLES. This collection is assigned the number *Six*. This process leads to *counting*.

And here is where the answer to your question lies. We can go on counting from now to eternity. We can go on writing down numbers until we use up all the pencils, all the paper. We can go on counting until we get tired, worn out and exhausted. But we can never come to an end. The stars will all burn out and die, and we will never come to an end. There is no number which can tell us "Here is the end, there is nothing after me."

We can count all the stars in the Universe. We can count all the atoms and all the electrons in the Universe; we may obtain a number which would tell us how many of them there are (approximately, however). But the number which we obtain, and we can obtain this number, is comparatively small.

As soon as we say that some number is, as you put it, the "highest actual number known to man," we can add unity to it (that is, go on counting) and we get a larger number.

As soon as mathematicians came to the realization that "there is no highest number known to man," they began to ask some very pointed questions. One of these was concerned with the problem of *the collection of all the numbers*. The mathematicians now say that such a collection consists of an infinity of members. And mathematicians assign a special number to such a collection. You see, Tom, if a mathematician is stumped, he does not get discouraged, he does not lose faith, he does not give up. If something is lacking, he invents something which fills the empty space, so that he can go on working.

So, young man, do not lose faith. You are young. Go on and study. By and by you will learn how to answer your own questions.

If you want to learn about very large numbers and about very small numbers, go to the Chicago Public Library and ask for my book, *Mathematics, Its Magic and Mastery*. You will find some interesting examples of such numbers. You may find a copy of this book in your school library.

Very sincerely yours,
AARON BAKST

Mathematics Department
STATE TEACHERS COLLEGE
MILLERSVILLE, PA.
April 14, 1952

DEAR DR. JONES:

I read your invitation for answers to the letter from Tom Hawk. Instead of Tom's trying to find the "highest actual number known to man" why doesn't he concentrate on gathering 15 or 20 very large (from the standpoint of number of digits used) numbers that man has actually computed or used in one way or another in actual life activities. I'll suggest four such numbers:

1. In my book *An Introduction to Mathematics for Teachers* (Henry Holt & Co.) I give an 80-digit number and a reference to its use by Sir Arthur Eddington in calculating the number of protons in the universe.

2. In the Dec. 17, 1951 issue of *Life* magazine, page 65, he can find a 42-digit number which tells the number of color patterns that are possible in a certain New York building entrance's electric sign.

3 and 4. In the January 1950 issue of *Mathematical Tables and Other Aids to Computation* are found calculated values for e and π to more than 2000 decimal places.

If Tom does not care to reproduce the 15 largest numbers in use that he has found, he might supply a bibliography. And, if he wants to correlate his study of Latin with that of mathematics he might tell us how to read several of the largest ones.

Sincerely yours,
LEE E. BOYER

H. T. DAVIS of Northwestern University pointed out that H. G. Hardy, in his book *Ramanujan* about his famous contemporary, the Hindu mathematician of that name, cites Skewes' number $10^{10^{10^{34}}}$ as what he believes to be the largest number which has ever served a particular purpose in mathematics. It was used in disproving a conjecture about the number of primes less than a given number N. Hardy also refers to Eddington's number, 10^{80}, and says that the number of possible games of chess is still larger, perhaps $10^{10^{50}}$.

In *Miscellanea 12* [THE MATHEMATICS TEACHER, XLIII (December 1950), pp. 418–419] we printed some discussion of big numbers and cited several references. There are now several additions to these notes. We discussed Mersenne numbers, $M_n = 2^n - 1$ at that time. Since then J. C. P. Miller and D. J. Wheeler of Cambridge, England have announced that they have devised a routine for testing the primality of numbers of the form $k \cdot M_{127} + 1$ on their electronic computer, *Edsac.* They have found more than ten primes greater than M_{127} which had been the largest known prime for approximately 75 years. As of July 1951, $180(2^{127}-1)^2+1$ became the largest known prime. In the same month A. Ferrier using a desk machine showed the primality of $(2^{148}+1)/17$ which was then the second largest known prime.[1] He had earlier shown the larger number $(2^{151}+1)/3$ to be composite.[2] *The Scientific American* calculated that it would take $1{,}809{,}250 \times 10^{69}$ columns of its magazine to print Miller and Wheeler's largest prime in their standard size type.[3]

Finally, on January 30, 1952, SWAC, in California programmed by R. M. Robinson, discovered the primes, $2^{521}-1$, and $2^{607}-1$, which lead to two new perfect numbers, the 13th and 14th to be known.[4]

Further discussion of Mersenne numbers, the related Fermat numbers, $F_n = 2^n + 1$, their history and relationships to the problems of perfect numbers and constructable polygons may be found in F. Klein, *Famous Problems of Elementary Geometry*, Second edition revised and enlarged by Raymond Clare Archibald (New York: Hafner Publishing Co., 1950), pages 81–85 especially, and in Oystein Ore, *Number Theory and Its History*, (New York: McGraw-Hill Book Co., 1948) pages 69–75. Page 75 of the latter begins a discussion of primes and their distribution which is related and of interest here since these problems gave rise to Skewes' number mentioned above.

As to number names, the English words representing the largest numbers having names which the editor of *Miscellanea* has been able to locate are *vigintillion* and

[1] *Nature*, CLXVIII (Nov. 10, 1951), 838.
[2] *Mathematical Tables and Other Aids to Computation*, V (January 1951), 55.
[3] *Scientific American*, CLXXXVI (February 1952), 40.
[4] *Mathematical Tables and Other Aids to Computation*, VI (January 1952), 61.

centillion. 10^{63} and 10^{303} are the numbers corresponding to these words in the American-French system in which a *million* represents 10^6; a billion, 10^9; a trillion, 10^{12}; etc.

George Soulés' *Philosophic Practical Mathematics* (5th edn. published by the author in New Orleans, 1905) suggests that the gap between vigintillions and centillions may be filled in by additional Latin forms as: *primo-vigintillion* for 10^{66}, *secundo-vigintillion* for 10^{69}, *tertio-vigintillion* for 10^{72}, etc.[5]

As noted in *Miscellanea 12* in the English-German system the number names and periods after a *million* progress by multiples of a million (10^6) rather than by multiplies of a thousand (10^3) as in the French-American system. Thus a *billion* is a million million ($1{,}000{,}000 \times 1{,}000{,}000 = 10^6 \times 10^6 = 1{,}000{,}000{,}000{,}000$) in England rather than a thousand million ($1{,}000 \times 1{,}000{,}000 = 10^3 \times 10^6 = 1{,}000{,}000{,}000$) as in this country. This difference, only for *trillions*, was noted semi-humorously in the article "Trillionism" in *Fortune* magazine, volume 42 (December 1950), page 74. Another reference to big numbers and their meaning, especially in public finance, is to be found on page 94 of *The Reader's Digest* volume 60 (January 1952).

E. P. Northrop in his *Riddles in Mathematics*[6] gets involved in some big numbers when he tries to answer the question "what can you do with four *2's*?"

Written in our notation the largest number used by Archimedes in his *Sand Reckoner* was 10^{63}. He showed that this was greater than the number of grains of sand in the universe.[7]

But, returning to the question of number names, Archimedes needed as new number words, in his scheme only the Greek equivalents of *octad*, *period*, and *order*, just as we can write and read enormous numbers using only exponential notation and terminology. The Hindus, too, were fascinated by big numbers and impelled by their custom of writing poetically to have names (and even several names) for each place of their decimal number system. For example, in Jaina works circa 100 B.C. *koti* meant hundred-hundred-thousand (10,000,000), a hundred-hundred-thousand koti was called *pakoti*, and so on up went the names to *asankhyeya* which in our notation would be 10^{140}. At this time the *Anuyogadvara-sutra* stated that the number of human beings in the world expressed in koti-koti would be represented by a number occupying twenty-nine places! One Jaina work included and named a period of time which would have required 194 places if written out.[8]

Big numbers seem always to be fascinating. Perhaps someone else can add to this round up.—P.S.J.

61. Another Unanswered Letter.

The following letter suggests one of many inconsistencies and/or inaccuracies that surprisingly enough creep into elementary mathematics. What is your answer to this? What other questions of this type have you observed?

1500 Strong Avenue
ELKART, INDIANA
March 26, 1952

Phillip S. Jones
Mathematics Department
University of Michigan
Ann Arbor, Michigan

DEAR SIR:

I am a junior in Elkart High School and am taking solid geometry. Our class is now studying cylinders and we seem to have run into a contradiction in definitions.

Given a right circular cylinder such as a mailing tube. If this tube is severed near the middle by a plane passing obliquely through it and if the parts are interchanged by placing

(*Continued on page 532*)

[5] My attention was called to *vigintillions* and *centillions* by Ray G. Kissinger and the reference librarians of the University of Michigan.

[6] E. P. Northrop, *Riddles in Mathematics* (New York: D. Van Nostrand Co., Inc., 1944), Chap. III, especially page 22.

[7] Sir T. L. Heath, *A Manual of Greek Mathematics* (Oxford, 1931), pp. 19–20, 327–330.

[8] B. Datta and A. N. Singh, *History of Hindu Mathematics* (Lahore: 1935), Part I, pp. 9–12.

RESEARCH IN MATHEMATICS EDUCATION

Edited by JOHN J. KINSELLA
School of Education, New York University, New York, N. Y.

The Question: To what extent can the ability to interpret data be improved through the teaching of elementary algebra?

The Study: Jackson, William N. *The Role of Algebra in the Development of Relational Thinking.* Ph.D. dissert. The Ohio State University, 1952.

BELIEVING that the ability to interpret data is an important aspect of the "relational thinking," emphasized in the Fifteenth Yearbook of the National Council of Teachers of Mathematics, Dr. Jackson sought to determine the extent to which the thirty-four students in two ninth-grade classes in the William Grant High School of Covington, Kentucky, could "develop understandings and abilities associated with the interpretation of data, through the study and use of algebraic concepts." (Page 1)

The investigator taught both classes and used similar methods and materials in each of them. The two sections were somewhat below national norms in reading and intelligence. The principal instrument used to evaluate growth in the ability to interpret data was the "Interpretation of Data Test, Lower Level," published by the Cooperative Test Division of the Educational Testing Service, Princeton, New Jersey. This test was administered in December 1950 and June 1951 and the gains evaluated for statistical significance.

Since the dominant teaching emphasis was on developing the ability to interpret data and the ability to reach conclusions inductively with other objectives playing a subsidiary role, this course varied considerably from common practice in teaching ninth-grade algebra. "The Language of Algebra," "Extending the Number System," "Approximate and Functional Relationships," and "Extending Operations in the Number System" were the four units constituting the year's work. The choice of these was greatly influenced by the investigator's belief that mathematical instruction, at least in elementary and secondary schools, should concentrate on the continuous development of the large concepts of number, measurement, relationship, symbolism, operation and proof. Although the content of the four units involved the usual formula, equations and graphs, these were used as instruments for analyzing and interpreting data rather than as ends in themselves. Furthermore, there was exceptional attention given to the study of linear and quadratic relationships through the concepts of differences, rates of change, slope and minimizing error squares. However, the greatest divergence of the course from the conventional one is a reflection of the investigator's concern for the "generalization of these understandings and abilities for use behind the narrow confines of elementary algebra." (Page 1) For instance, the teaching materials on symbolism involved more than the interpretation of the mathematical type; it included such items as the meaning of "a red light," "the sound of a siren," a flag, highway signs, and abbreviations. The notion of variable was extended from the domain of number to the variations of the meaning of a word in different sentences and contexts and to statements of dependence among qualitative factors, such as color and type of work. The notion of oppositeness usually associated with positive and negative numbers was extended to synonyms and

antonyms and to opposite operations in general. Symmetric and reflexive relations were not confined to algebraic and arithmetic equalities and substitutions but were applied to the analysis of propositional statements and converses in a variety of non-mathematical situations. Similarly, the notion of transitive relations was treated. It seems obvious to this reviewer that procedures like these are similar to those used in the "nature of proof" experiments of Fawcett, Ulmer, Gadske and Lewis. Since these were highly successful in the geometric atmosphere, it does not seem ridiculous to test an analogous hypothesis in the algebraic scene.

The primary teaching materials were study guides; textbooks were available in the classroom for occasional reference and comparison only. The study guides consisted of questions and problems arranged in order of increasing difficulty and leading from specific to more general statements. Every guide contained a final summarizing question. Usually, only one principle, or part of one, was developed in each study guide, which was of such a nature that it could be completed in one class period. These guides were made part of, and supplemented by, class discussions, group presentations by the teacher, and supervised study procedures in which the teacher gave individual guidance. In individual folders the students filed their completed study guides and kept a record of the generalizations or class agreements arising out of the teaching-learning procedures.

Some of the findings of this study follow:

1. Statistically significant gains were made by both groups in (a) general accuracy when interpreting data, (b) ability to evaluate statements involving insufficient data and (c) ability to evaluate statements which involve going beyond the limits of the data. The test used sought to measure the ability to perceive relationships in data by making comparisons, to see common elements in data, to recognize prevailing tendencies in the data, to evaluate the ability to recognize the limitations of data by detecting the need for additional information, and to suspend judgment until the necessary facts are on hand.
2. More significant gains were made by the more intelligent portion of the student group although the gains of both groups, regardless of intelligence level, were statistically significant with respect to items 1a, 1b and 1c above.
3. However, strange as it may seem, the poorer reading group gained more than the better reading group with respect to 1a and 1b above. This is partly explained by the fact that seven of the twenty poorer readers were not in the low intelligence group.
4. The boys did not gain as much as the girls although they were superior to them in their scores on the tests of intelligence and reading.

If it be granted that the ability to interpret data is especially important in a democracy which depends upon the individual judgments of the electorate, then it seems that this study implies, or at least suggests, that teachers of mathematics have an opportunity to make a significant contribution to our future citizens, irrespective of their present status with respect to intelligence or reading ability.

Mathematical Miscellanea

(*Continued from page 530*)

the bases adjacent to each other, is the cylinder thus formed circular or elliptical?

On page 62 in James' and James' *Mathematics Dictionary* a circular cylinder is defined as a cylinder having a circular base and an elliptical cylinder as having an elliptical base. Using these definitions, the cylinder is elliptical. However, on page 63 of the same book, it says a cylinder is defined by its right section. If this is the case, the cylinder is circular. On page 62 a right circular cylinder is defined as a cylinder whose right sections are circles. So depending on the definition used, the cylinder can be elliptical, circular, or right circular. We have thrown out the last possibility as being unreasonable. However, we are still confused about the other two.

Our teacher holds the cylinder to be circular. He says cylinders are defined with the supposition that the directrix is perpendicular to the base. We "rebels," however, are still doubtful.

Please, can you help us?

Sincerely yours,
DON MARQUIS

The President's Page

PLANS FOR NEW PUBLICATIONS AND FOR PUBLICITY

IT IS MY PLAN from time to time to report on this page special activities of the National Council. Many advantages can come to our organization as the membership keeps informed about programs and studies in progress and for which in many cases the members can make valuable suggestions. Suggestions will be welcomed by those responsible for the activities or by me. This month brief preliminary reports will be made on the work of two important committees.

Committee on Publications of Current Interest. At Des Moines, recommendations from a new Committee called the Committee on Publications of Current Interest (CPCI) were approved. The purpose of the Committee is to provide inexpensive publications concerning the teaching of mathematics in grades one through twelve. It is intended that the publications should all sell for less than $1.00 and that many should be in the 5¢ to 25¢ bracket. These publications will be less permanent in importance than yearbooks, more specialized in appeal than articles in THE MATHEMATICS TEACHER, and will contain materials which teachers may wish to purchase in quantity for their own discussion groups or for their pupils to use.

That topics suggested for publications, sponsored by this Committee, cover a wide range, is illustrated by such titles as: Using History in the Teaching of Mathematics, Score Sheet for Evaluating Textbooks, Mathematical Field Trips, List of Courses of Study, Homework for Parents, How to Locate the Decimal Point, How the Mathematics Teacher and the Librarian Can Work Together.

Members of the Committee are:

M. H. Ahrendt, Washington, D. C.; Charles Butler, Kalamazoo, Mich.; Hope Chipman, Ann Arbor, Mich.; Janet Height, Wakefield, Mass.; Donovan A. Johnson, Minneapolis, Minn.; Joy Mahachek, Indiana, Pa.; H. Vernon Price, Iowa City, Iowa; Henry Swain, Winnetka, Ill.; and Henry W. Syer, Boston, Mass., Chairman.

Publicity Committee. Mathematics teachers have recognized for some time that general publicity on the advantages of the study of mathematics has been inadequate. A Publicity Committee was appointed two years ago by former President Harry Charlesworth to assist in the correction of this situation. The Committee has recommended to the Board of Directors that special groups be appointed to assume responsibilities for various phases of desirable publicity in this area. These smaller groups will be at work on this problem during this school year with responsibility for co-ordination assigned to the Publicity Committee of which Kenneth Brown, mathematics specialist in the U. S. Office of Education, is chairman. Other members of the Committee are:

M. H. Ahrendt, Washington D. C.; Allene Archer, Richmond, Virginia; H. G. Ayre, Macomb, Illinois; Ida Mae Heard, Lafayette, Louisiana; Mabel Simcox, Chicago, Illinois; Gilbert Ulmer, Lawrence, Kansas.

The Committee hopes to promote the publication of articles showing the importance of mathematics in effective living in our society, articles on the value of study of mathematics in schools, and articles on the teaching of mathematics and on the role that The National Council of Teachers of Mathematics is playing in the improvement of mathematics education. It is hoped that articles of this nature will soon appear in appropriate state and local education journals, in national journals of education, and in popular magazines.

JOHN R. MAYOR
President

WHAT IS GOING ON IN YOUR SCHOOL?

Edited by

JOHN A. BROWN
Wisconsin High School
Madison, Wisconsin

and

HOUSTON T. KARNES
Louisiana State University
Baton Rouge, Louisiana

CURRICULUM REVISION AT MOLINE

This article is being written by a group of mathematics teachers who double as "grass-roots" curriculum workers. It is an attempt to encourage mathematics teachers who are not satisfied with their present curriculum to do something about it. We hope that people who are contemplating a curriculum revision program in mathematics may profit from this explanation and analytical evaluation of a mathematics curriculum revision program that was started in the thinking of its instigators some five years ago. The program, as it has developed, has been in practice in the classroom for the last three school years.

The University of Chicago conducted a survey of the Moline public schools in the spring of 1946. Leading out of this survey was an investigation by the secondary school mathematics teachers of the offerings in the field of mathematics at that time. The investigation revealed a lack of uniformity in structure, objectives, materials, and standards in our secondary schools; there were also a great many drop outs during the junior high school period. Questionnaires were sent to the entire faculty of the secondary schools asking their opinions of the value of the present mathematics program and which topics needed stressing with respect to their departments. The mathematics teachers were also given an opportunity through a questionnaire to express their opinions on further study of this problem. It was not compulsory that every mathematics teacher work in this program and we did not get 100% cooperation at the start. It is a pleasure to say that now, five years later, we do have every mathematics teacher active in the revision program.

In 1949, the University of Illinois in cooperation with the State Department of Education gave us the opportunity to become part of the state-wide curriculum program. We were fortunate to obtain the advice and assistance of Dr. D. W. Snader of the University of Illinois, Dr. E. W. Hellmich of Northern Illinois State Teachers College, Miss Lenore John of the University of Chicago Laboratory School, and Mr. Roy Clark of the State Department.

Working with these advisors the committee decided on a revision program which was called the "broad front" approach to mathematics. The program is planned to be "broad" in the following four aspects:

1. It involves the entire community.
2. It involves cutting across the subject lines of traditional mathematics on every grade level and giving the child the most mathematics he can handle at each level. Each year contact will be made with all areas of mathematics, general emphasis being placed on particular areas such as arithmetic in grade 8, and algebra in grade 9.
3. The difference between students' work lies in the extent to which they are capable of doing work.
4. It should not be so broad that it has no depth.

Since this is written to aid and abet those teachers who are working or who are interested in beginning work on curriculum problems, it may be profitable to include some of the problems that confronted the committee.

Like most other worth-while things, the value of the program must be shown before it will gain recognition and support. Thus for the first year a committee of five people had to lay the ground work for the program entirely on their own time; this meant working after school and evenings. The next year the Board of Education felt the program warranted school time for planning so the allotted six teachers each one hour a day. This proved to be a very expensive method and was discontinued the next year in favor of all-day meetings whenever the committee felt the need to work together. By this time much of the work had been delegated and so it was necessary to meet as a group only occasionally. Two or three times a year the advisors would meet with the committee to discuss the progress at that time. The latest plan for working has been to take a block of time such as two weeks and work in groups of two's for two or three days while substitutes are hired to take the classes. At the end of the time the group meets together to coordinate the material and put it into usable form. This program did involve some extra work for a teacher since it meant leaving the class room for a few days. The committee is still using this method and supplementing it with after-school and evening meetings.

In the fall of 1949, the program was put into operation at the seventh grade level and each succeeding year one more year of the program has been started. At the present time we are operating through the ninth grade with a tenth grade program ready for next year. One of the problems involved here has been the planning of each new year while evaluating and revising the program being taught.

Another problem involved in such a program is the means of evaluation. How do you know your program is succeeding? We did not have a control group at any time so we had nothing with which to compare and we had no achievement test grades of previous seventh grade groups. The available standardized tests did not completely measure what we had set up as our objectives.

At the present time our evaluation program consists of standardized tests being given at the beginning of grade seven and near the end of grades seven, eight, and nine. The progress of each pupil is then plotted and compared with the normal expected growth. During the past year a committee has started work on a test which is to supplement the standardized tests. It is planned that this test when put into use can be used to measure progress toward the objectives set forth for the program, while the standardized test will continue to be used. The question "Is the new program really doing the job?" still cannot be measured objectively.

Many of you may wonder if the school has done any public relations work and the answer is "yes." The Board of Education invited us in to present our plans and to question us concerning the program. At P.T.A. meetings in the junior high schools, there was a period for questions and answers and this proved most successful. Although a great number of the parents were contacted through our P.T.A. more needs to be done to inform them of our work.

Business and industry were invited to participate in a panel discussion at one of the general faculty meetings. Business, industry, and the schools have taken a step forward in promoting better relations with a Business, Industry, Education Day on which educators visit the industries and businesses and become better acquainted with their community. During alternating years people from business and industry visit the schools and learn first-hand what is going on here. The entire mathematics staff took an additional day to visit various industries in attempts to determine the mathematical needs of the community.

In writing this article we hope that the problems presented concerning "grass-

roots" curriculum planning will not leave you with a feeling that there are not satisfying experiences. On the contrary, the five years we have spent working on this program have probably enriched the experiences and contributed to the growth of the participating teachers more than any other experience could. We are grateful for the opportunity to teach in a school system that allows their teachers of mathematics to be in on the planning of the program they teach. This is only a brief survey of some of our experiences but we hope it will encourage all of those mathematics teachers who feel their mathematics program is not completely satisfactory to get their revision program under way.

From *The Illinois Council of Teachers of Mathematics News Letter*, May 1952, by JANE JAEGER and JACK MERWIN for the Committee

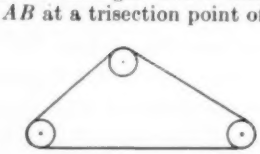
LOS ANGELES CITY COLLEGE MATHEMATICS PRIZE COMPETITION

During May, 1952, Los Angeles City College held its second annual William B. Orange Mathematics Prize Competition for high school students of Los Angeles City High Schools.[4] Thirty-five schools participated with thirty-two schools entering full teams. Each school could not enter more than five students. The three students making the highest score were considered the team representing the school. This year Washington High School won the team prize which is the William B. Orange Mathematics Competition Perpetual Trophy. Seven individual prizes consisting of cash, slide rules, books and professional magazine subscriptions were awarded to students from the following high schools: Hamilton, Wilson, Washington, Marshall, Van Nuys, Manual Arts, and Roosevelt. Ten additional students were awarded mathematics handbooks and nine more were listed for honorable mention. Questions used for the competition were:

Time: 2 hours

Show all work on problems. Partial credit will be given for partial solutions.

1. (5%) Factor completely x^8+x^4+1.
2. (5%) If three machines can make six articles in nine minutes, how long will it take 100 machines to make 300 articles?
3. (5%) A man drives from Los Angeles to San Francisco at the rate of 30 miles per hour and returns over exactly the same route at the rate of 60 miles per hour. What is his average speed for the round trip? Neglect starting, stopping, turning, etc.
4. (5%) Insert two positive numbers between 5 and $13\frac{1}{3}$ so that a sequence of four numbers is formed with the first three in arithmetic progression and the last three in geometric progression.
5. (5%) Solve for y in terms of x: $2 \log_{10} x-5 = \log_{10} y$.
6. (5%) E is the midpoint of side AB of equilateral triangle ABC. D is on the opposite side of AB from C and ADE is an equilateral triangle. Prove that CD intersects AB at a trisection point of AB.



7. (5%) A thin continuous belt is stretched taut around 3 pulleys each 2 feet in diameter, as shown. The distances between the centers of the pulleys are 6, 9, and 13 feet. What is the length of the belt?
8. (5%) Find the smallest number that if divided by any integer from one to six gives a remainder one less than the divisor.
9. (5%) Pole A is 3 feet tall and 20 feet from pole B which is 12 feet tall. A wire is to be stretched from the top of pole A to a point on the ground and thence to the top of pole B. What is the shortest possible length of wire?
10. (5%) Show that $x^2+y^2+z^2 \geqq xy+yz+xz$ for all real values of x, y, and z.
11. (10%) Given $y=mx+3$ and $x^2-y^2=3$.
    (a) Solve for x in terms of m.
    (b) From your answer in (a) find the values of m for which the x's are equal and determine the x's if they are defined.
12. (10%) A twenty inch wire is bent to form an isosceles triangle with the base a double thickness of wire. Express the area of the triangle in terms of one of the equal sides.
13. (10%) Show that $x^5/5+x^3/3+7x/15$ is an integer for every value of x which is an integer.

(*Continued on page 542*)

[4] The first annual competition was reported in "What's Going on in Your School?," THE MATHEMATICS TEACHER, XLV, No. 1 (January, 1952) pages 34 and 35.

AIDS TO TEACHING

Edited by

Henry W. Syer
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Boston, Massachusetts

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EQUIPMENT

E. 91–E. 117—Working With Numbers Teaching Aids

E. 91—Picture-Symbol Cards (1A1)

E. 92—Group Recognition Cards (1B1)

E. 93—Group Recognition Cards (1B17)

E. 94—One-More Cards (1C1)

E. 95—Partial Counting Cards (1D1)

E. 96—The 100-Chart (1E1)

E. 97—Single Facts Cards (1G1)

E. 98—Related Facts Cards (1H1)

E. 99—Readiness Test Material

E. 100—Teacher's Instructions for First Grade

E. 101—Fraction Cards (2A1)

E. 102—Counting-by-2's Cards (2B1)

E. 103—Partial Counting Cards (2D1)

E. 104—The 200-Chart (2E1)

E. 105—Single Facts Cards (2G1)

E. 106—Related Facts Cards (2H1)

E. 107—Teacher's Instructions for Second Grade

E. 108—Single Addition Facts Flash Cards (3A1)

E. 109—Single Subtraction Facts Flash Cards (3S1)

E. 110—Single Multiplication Facts Flash Cards (3M1)

E. 111—Single Division Facts Flash Cards (3D1)

E. 112—Teacher's Instructions for Flash Cards

E. 113—Multiplication and Division Cards (3F1)

E. 114—Teacher's Instructions for Multiplication and Division Cards

E. 115—Fraction Cards (3F2)

E. 116—Fraction Cards (3F3)

E. 117—Teacher's Instructions for Fraction Cards

The Steck Company, Publishers, Austin 61, Texas.

Arithmetic aids; see below for descriptions and prices. This complete set of materials in color is intended to be sufficient for the work in grades one through three. Each item is described below with the price following the identification number. The number which should be used to order these items from the company is given in the list above after each item there. It is to be noted that E. 90 through E. 99 are meant to be a complete set for the first grade and sell for $5.75; E. 101 through E. 106 are for the second grade and sell for $6.25; and the other items are for grades three and above.

Description of E. 91: ($.50) There are 10 cards (7″×11″) containing groups of objects and a number ($1\frac{1}{4}''$ high) on each card. The numbers 1 through 10 are used.

Description of E. 92: ($.50) There are 16 cards (5″×5″) with pictures of objects only in groups; 1 group of 1, and 3 groups each of 2 through 6.

Description of E. 93: ($.50) There are 16 cards (5″×5″) with groups of circles; 1 group of 1 and 3 groups each of 2 through 6.

Description of E. 94: ($.50) There are 15 cards (7″×4″) with pictures of objects, 3 each to illustrate 2+1, 3+1, 4+1, 5+1, and 6+1.

Description of E. 95: ($.50) There are 35 cards (7″×4″) with a line down the center of the cards and $\frac{3}{4}$″ circles on each side to illustrate such facts as 2+1, 3+3, 5+2, 6+4, etc.

Description of E. 96: ($1.50) This set consists of 10 strips (3″×22″) with 10 dots (1″) on each, 12 single dot cards (2″×3″), and 128 number cards (3″×3″) with the numbers 1 through 100 and some duplicates.

Description of E.97: ($.50) There are 30 cards ($3\frac{1}{2}$″×$5\frac{1}{2}$″) with 15 addition facts and 15 subtraction facts in vertical arrangement. One side contains the problem without the answer, the other side with the answer.

Description of E. 98: ($.50) Each of these 12 cards ($5\frac{1}{2}$″×7″) contains two addition or two subtraction facts (6 cards of each). The addition facts combine the same two numbers and have the answer on the back; the subtraction cards. combine such facts as

$$\begin{array}{r} 5 \\ -1 \\ \hline \end{array} \qquad \begin{array}{r} 5 \\ -4 \\ \hline \end{array}$$

Description of E.99: ($.75) There are 7 cards(5″×5″) with groups of objects; 8 cards (4″×4″) with groups of circles; and 4 cards (3″×3″) with abstract numbers. There are also class record blanks to help in tabulating the results of determining the degree of readiness which members of the class have for number work.

Description of E. 100: ($.25) The teachers manual is $5\frac{1}{2}$″×$8\frac{1}{4}$″ and contains 24 pages. It is illustrated with pictures of the material and takes up each set of cards separately.

Description of E. 101: ($.50) There are 10 cards ($5\frac{1}{2}$″×6″) showing circles, squares, rectangles and triangles divided and colored so as to illustrate $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$.

Description of E. 102: ($.50) There are 25 cards (2″×2″) containing the even numbers from 2 through 50. They are printed on a slightly different colored paper so that they will contrast with the odd numbers on other cards, but the contrast is not strong enough and the point of "even versus odd" numbers is far from clear.

Description of E. 103: ($.50) Here are 33 cards (7″×4″) divided down the center by a line with a set of dots on each side to aid in forming such number combinations as 4+3, 2+1, 3+2, 3+1, 4+4, 5+3, 4+1, etc.

Description of E. 104: ($2.50) This set consists of a chart (22″×22″) with 100 dots on it, 10 strips (3″×22″) with 10 dots each, 12 cards (2″×3″) with one dot each, and 238 cards (3″×3″) with numbers on them.

Description of E. 105: ($1.25) Here are 120 cards ($3\frac{1}{2}$″×$5\frac{1}{2}$″) with 60 addition facts and 60 subtraction facts. The problem without answers is on one side and with answers on the other.

Description of E. 106: ($1.00) Here are 54 more related facts cards ($5\frac{1}{2}$″×7″); 27 on addition and 27 on subtraction.

Description of E. 107: ($.25) This teachers manual is $5\frac{1}{2}$″×$8\frac{1}{4}$″ and contains 16 pages with illustrations.

Description of E. 108 through E.111: ($1.25) each) The addition, subtraction and multiplication sets contain 100 cards ($3\frac{1}{2}$″×$5\frac{1}{2}$″) each, and the division set contains 90 cards. One side is without answers and the other side with answers.

Description of E. 112: ($.25) This manual is $5\frac{1}{2}$″×$6\frac{1}{4}$″ and contains 19 pages.

Description of E. 113: ($1.25) In this set 25 cards (9″×12″) show such semi-concrete figures as circles, squares, hearts and rectangles arranged in groups of groups. For example, one card has four groups of three hearts each to build the concepts of 4×3 and 12÷4.

Description of E. 114: (\$.25) This manual is $5\frac{1}{2}'' \times 8\frac{1}{4}''$ and contains 13 pages.

Description of E. 115: (\$1.25) These 50 cards ($4'' \times 9''$) show sets of circles, squares and rectangles in groups colored in different ways to illustrate the fractional parts of the group. It should be noted that poor combinations of colors are sometimes used for students who are red-green blind.

Description of E. 116: (\$1.25) These 50 cards ($5\frac{1}{2}'' \times 6''$) have fractional parts of such figures as circles and squares colored in to show fractional parts of a whole.

Description of E. 117: (\$.25) This manual is $5\frac{1}{2}'' \times 8\frac{1}{4}''$ and contains 14 pages. It discusses both E. 115 and E. 116.

Appraisal of E. 91 through E. 117: This whole set of materials is attractive and planned with a great deal of attention to its practicality. Each teacher may not agree with all the methods of presentation suggested in the manuals, but there are enough new ideas to make them very valuable. The cards and charts are so basic that they can be adapted to any teacher's methods of presentation. Even if some are not used, or some of a teacher's favorite devices are not included, it would be very wise to have this material among the teaching aids of every elementary school.

In general the material used in construction is quite sturdy and should stand up under constant usage. The colors are attractive, but not always bright enough nor, as noted above, chosen for ease of distinguishing features which are important. Such minor lapses as this detract very little from the value of this whole set. It is sometimes easier to obtain permission to buy a complete set of materials of this magnitude in a school than it is to get approval of many small orders for the component parts.

E. 118—The Magic Teacher Puzzle-Plans

Follett Publishing Company, 1257 S. Wabash Ave., Chicago 5, Illinois or 381 Fourth Ave., New York 6, N. Y.

Puzzles; Sets of cards; \$1.00 each with discount for quantity orders.

Description: Puzzle-Plans consist of 5 sets of jig-saw puzzles. Two deal with Basic Addition Combinations, two with Basic Subtraction Combinations, and one with Number Concepts and Recognition. The first addition set (NA1) consists of 21 cards. Each card has a basic addition combination printed on the upper or white portion, and has the answer or the sum of this combination on the lower or colored portion of the card. The card is die cut to separate the problem from the answer. The second addition set (NA2) has 24 additional Basic Addition Combinations of slightly greater difficulty than the first set. The first subtraction set (NS1) has 23 Basic Subtraction Combinations and the second set (NS2) has 22 additional Basic Subtraction Combinations of slightly greater difficulty than the first set. Each card in these four sets is made of stiff cardboard measuring $2\frac{1}{2}''$ by $4''$. Since the cards are cut with different matching patterns, the problems and answers cannot be incorrectly matched.

The fifth set (NRC) is designed to teach Number Recognition and Number Concept using numbers from 1 to 12, inclusive. This set consists of five cards each $5''$ by $8''$ in size. Four of these cards have a red background and the fifth card is white. Each of the four red cards is divided into three sections making a total of 15 sections. Each section carries, on the left-hand side, a picture designating a certain number of objects. These numbers begin with 1 and continue through 12. The corresponding numeral is on the right-hand side of the section. Each numeral is printed on a die-cut, punch-out block, and the blocks are designated with different patterns. Since the die-cut blocks are of different shapes and sizes, the child can correctly fit them only in the proper places. Thus the numerals must be correctly matched with the corresponding number of objects. The fifth or white card in the set is a test card. Each of the 12 sections

has a certain number of dots representing the numbers from 1 through 12. The numbers, themselves, are carried on small punch-out blocks. These punch-out blocks are all the same shape and size and must be re-assembled by knowledge of the numbers.

Appraisal: The use of this set of puzzles is in line with the modern trend in using devices and games in teaching numbers. One advantage that this particular set of games offers is that the child does not have to know the answer in advance. By finding the lower section which fits an upper section of a card, the child has the answer. By repetition of the game, the child learns the answers to the various combinations while having fun. This is an attractive and fascinating set of puzzles which can be used to good advantage in the kindergarten, pre-primer and first grade as well as for remedial work in other grades. The teacher using these puzzles would need to supervise and follow up their use rather carefully since the game feature could very well offset the desired number experience. The addition and subtraction sets can be used separately at first and then together. There are other educational advantages in these games, but the outstanding feature is that the child learns as it plays.

FILMSTRIPS

FS. 138–FS. 143—Mathematics Series

FS. 138—Mathematics in Aviation: The Compass (32 fr.)

FS. 139—Mathematics in Aviation: Wind Drift (36 fr.)

FS. 140—Indirect Measurement (38 fr.)

FS. 141—Systems of Equations (30 fr.)

FS. 142—Slide Rule: Part 1 (24 fr.)

FS. 143—Slide Rule: Part 2 (27 fr.)

McGraw-Hill Book Company; 330 West 42d St., New York 36, N. Y.

B & W ($5.00 each or $27.00 for set of 6)

Description of FS. 138: The compass is described as a substitute for a sign post in the sky so that the aviator will be able to reach his destination. The basic principles of a compass are shown by picturing the construction of a compass and showing its relation to the earth's magnetic field. It explains how the pilot uses the compass to determine the course he is flying and gives detailed directions on how to correct for magnetic variation.

Appraisal of FS. 138: A simple direct explanation of a device that requires simple geometry in its operation. It will give a realistic application of mathematics in a glamorous occupation. Thus, it is suitable to motivate the junior high school mathematics class as well as furnish a base for problems in reading and computation with angles.

Description of FS. 139: The meaning of wind drift is illustrated by a balloon and an airplane and then explained as the difference between a plane's true heading and its true course over the ground. The operation of a drift meter is explained and the use of drift indicator to determine wind direction and wind speed is demonstrated. This discussion involves simple scale drawings to determine true heading and wind speed. The strip ends with a discussion of vectors and their importance to pilots and navigators.

Appraisal of FS. 139: This filmstrip gives a simple, direct explanation of wind drift which requires the application of mathematics in a realistic situation. Although an introduction is given to vectors, the strip is suitable for junior high school mathematics classes. It will probably be most useful in introducing a unit on scale drawing.

Description of FS. 140: After showing some places where indirect measurement is needed, the strip works out step-by-step a typical surveyor's land measurement problem. To determine the boundaries of a farm, the surveyor first makes a rough scale drawing and then goes into the field to make measurements. From one known station, he makes measurements with transit and steel tape. The

strip also demonstrates the use of trigonometry in taking sights around obstacles by solving the hypotenuse of a right triangle to determine an unknown distance.

Appraisal of FS. 140: This strip should be very useful for introducing a unit on indirect measurement or for preparation for field work. A practical problem is used which introduces a variety of techniques used in surveying. Some technical terms such as bearing, hypotenuse, and sine 45° are used.

Description of FS. 141: This filmstrip makes the solution of a system of equations practical by illustrating the interception of a ship at sea with a helicopter. The formula connecting time, rate and distance for the ship and helicopter illustrates the meaning of a common solution as well as the meaning of consistent, inconsistent, or depending equations. Illustrations are given to show the solution of systems of equations by addition, subtraction, or by the elimination of one unknown and solving for the other. Solution by substitution is worked out and a general method derived to be applied to similar problems.

Appraisal of FS. 141: Although this strip contains some typical textbook explanations, it will furnish the teacher with good applications of systems of equations as well as illustrate concretely the meaning of different types of systems. The drawings are appropriate and the captions adequate. The strip will probably be most useful in introducting systems of equations in the ninth grade.

Description of FS. 142: The construction of the slide rule, its component parts, and scales *A*, *B*, *C*, *D* are first explained. A simple multiplication example is illustrated to show the process and the method of placing the decimal point in the answer by estimation. Several multiplication problems are then set up for practice. Similarly division is illustrated with a brief discussion of the accuracy of slide rule computation.

Description of FS. 143: A problem in proportion is solved to illustrate the method of using the slide rule and the method of placing the decimal point by estimation. Similarly the *A* and *D* scales are used to find the square root of numbers. Several problems in square root are set up to show the method of placing the decimal point in the answer by estimation and by the grouping of figures.

Appraisal of FS. 142 and FS. 143: These filmstrips illustrate good techniques for teaching elementary operations with the slide rule. It is doubtful that they are more effective than a large model slide rule. They are as expensive as a model, more cumbersome to use and less versatile. It is unfortunate that the strips did not deal adequately with the problem of reading the slide rule scale. In the experience of the reviewer this is a source of real difficulty in teaching the slide rule. By the enlarging of sections of the scale a filmstrip could be considerably more effective in presenting this phase of instruction than a model slide rule.

FS. 144–FS. 152—Bridging the Decades

FS. 144—Review; Work and Play with Number 11

FS. 145—Work and Play with Numbers 12 and 13

FS. 146—Work and Play with Numbers 14 and 15

FS. 147—Work and Play with Numbers 16 and 17

FS. 148—Work and Play with Number 18

FS. 149—Work and Play with Number 19

FS. 150—Work and Play with Number 20

FS. 151—Work and Play with Problems

FS. 152—Work and Play with More Problems

Eye Gate House, Inc., 2716 Forty-First Ave., Long Island City, N. Y.

Color ($4.00 each, $25.00 for set); 25 frames each.

Description of FS. 144 through FS. 152: The first seven in this set gave illustrations of addition and subtraction facts involving the numbers 11 through 20. Each number is illustrated by groups of familiar objects such as rabbits, boats, balls, tops, and so on. Then there are summary frames giving the number facts as questions for the pupils to answer. Toward the last of these seven filmstrips some simple problem-situations begin to appear using menus and price lists to ask number questions.

The last two filmstrips contain "problem" material such as the following: counting by 2's, 3's, and 5's; adding 3 numbers with one digit each, with two digits each; subtracting two place numbers; the concepts of $\frac{1}{2}$, $\frac{1}{4}$, followed by questions such as "$\frac{1}{2}$ of 4," "$\frac{1}{4}$ of 8;" the concepts of *dozen* and *half-dozen*.

Appraisal of FS. 144 through FS. 152: The selection of groupings which have been chosen to represent the numbers seems to show care and an understanding of the variety of illustrations needed. The vivid and appealing colors add a great deal to material to be presented to the age for which this is intended. Of course this material could be drawn on the blackboard or printed in textbooks for pupils to see; then one would lose much of the color and the surety that the whole class were directing their attention to exactly the same material. These advantages must be weighed against the cost and mechanical difficulties of projection. Such material should not be condemned by a teacher without having given it a fair trial with an actual teaching situation. Some frames here are indispensable, some could better be done at the blackboard by a wise teacher.

The last two filmstrips have such a variety of types of materials, over a wide range of topics, that it will be difficult to know just where and how to use them. The "teacher's manual" supplied with these nine filmstrips is four pages long and is a compilation of trite educational expressions proving completely inadequate as an aid. In general, this set of teaching aids could be a tremendous help to any teacher who will use them with imagination.

MODELS

M.28—Doodler

Kenner Products Company, 912 Sycamore Street, Cincinnati 2, Ohio.

Flexible model; $1.00.

Description: Essentially the wire pieces of this device make a seven-sided figure out of 30 pieces of wire and 14 colored beads. It is 6″ in diameter when flat, but may be distorted into many shapes including a sphere ($4\frac{1}{2}$″ in diam.), and a cylinder (4″in diam., and $5\frac{1}{2}$″ in height). Of course, these shapes are not exact, but only approximate.

Appraisal: The figures into which this can be distorted are useful to illustrate the symmetric, beautiful side of intuitive geometry, but may also be used to suggest the computation of volumes which are unusual and not covered by the common formulas. Everyone who manipulates the device is fascinated with it. It probably contains more elements of art than mathematics but may serve as one small bridge between the two.

What Is Going On?

(*Continued from page 536*)

14. (10%) Three men A, B, and C, evenly spaced on a circular track in that order start walking around the track in the same direction, A walking toward B. A overtakes B in 15 minutes and overtakes C in 5 additional minutes. How soon does B overtake C?
15. (10%) $ABCDEF$ is a regular hexagon. The midpoints of sides AB, CD, DE, FA are joined respectively to points E, F, B, C, forming a small rhombus at the center of the hexagon. Express the area of this rhombus in terms of a side of the hexagon.

Report of the Competition and the questions used was submitted by

Ben K. Gold
Competition Committee Chairman,
Mathematics Department,
Los Angeles City College.

DEVICES FOR A MATHEMATICS LABORATORY

Edited by Emil J. Berger
Monroe High School, St. Paul, Minnesota

This section is being published as an avenue through which teachers of mathematics can share favorite learning aids. Readers are invited to send in descriptions and drawings of devices which they have found particularly helpful in their teaching experience. Send all communications concerning Devices for a Mathematics Laboratory to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

A Combination Lock

Three years ago while collecting materials for a unit on permutations, combinations, and probability in second course algebra we came across an illustration of a combination lock that was supposed to be in existence around 1560.[1] As part of the preparation for this particular unit of work one student was assigned the task of building a workable lock. Unfortunately technical difficulties prevailed and the student never completed the assignment. The following year two students accepted the challenge, but again the assignment proved to be too difficult. In the spring of 1952 the assignment was repeated once more. This time three students undertook the task. They hit the jackpot; all three completed workable locks based on combinations. One of the students produced an electrically operated gadget with three dials; the second student devised an intricate little machine with six plastic discs; and the third student completed a lock made entirely of wood. It is this last device that will be described in this article.

The short introductory résumé describing the successes and failures which different students experienced in developing the lock device is presented here as an illustration of the kind of thing that must be expected when students are asked to develop their own multi-sensory aids. Occasionally an idea may have to be discarded or postponed, but in the end a challenging project proposed with little or no direction from the teacher often helps arouse much more interest among students than do easier tasks completed with considerable teacher assistance. This was certainly the case with the combination lock. It was a topic of conversation for weeks both in and out of the classroom. Of interest too is the fact that prior to actual completion of the locks student discussions centered almost entirely around construction details, but following completion interest swung around to the mathematical principles involved.

The combination lock described in this article consists of five cylinders, a cradle shaped frame, and a U-shaped key and lock bar. Materials needed to build the device include one piece of clear pine $\frac{3}{4}'' \times 4'' \times 24''$, one piece of plywood $\frac{3}{8}'' \times 3'' \times 12''$, one round dowel $\frac{1}{2}'' \times 14''$, one round dowel $\frac{1}{4}'' \times 6''$, four thin 1″ wood screws, and a quantity of small household nails. The following directions will be found helpful in carrying out the details of construction:

Cut five cylinders $2\frac{3}{4}''$ in diameter with a jig-saw from the $\frac{3}{4}''$ pine board; the altitude of each cylinder will be the same as the thickness of the board. With a 1″ wood auger bore a hole 5/16″ deep at the center of one side of each cylinder (Fig. 1). Then drill a $\frac{1}{2}''$ hole through the remaining 7/16″ using the same center. The result of the two operations will be a $\frac{1}{2}''$ hole with a $\frac{1}{4}''$ circular shoulder (Fig. 2). Finally cut a keyway $\frac{1}{4}''$ wide off the $\frac{1}{2}''$ hole across the

[1] F. W. Kokomoor, *Mathematics in Human Affairs* (New York: Prentice-Hall, Inc., 1942), p. 488.

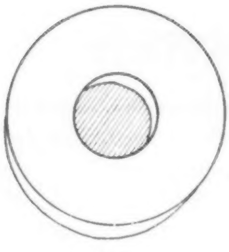

Fig. 1

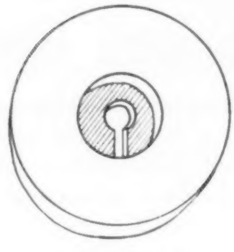

Fig. 2

shoulder to the edge of the 1″ bore (Fig 2).

To build the cradle shaped frame, three pieces are needed—a rectangular base, a left end piece, and a right end piece (Fig. 3). The base may be cut from clear pine; its dimensions are $\frac{3}{4}'' \times 2\frac{3}{4}'' \times 4''$. The two end pieces should be cut from $\frac{3}{8}''$ plywood according to the dimensions shown in the diagram of Figure 3. Do not use a substitute material, otherwise a lot of other dimensions will need to be changed later on.

The U-shaped key and lock bar is not difficult to construct but the work must be fairly accurate if the lock is to operate properly. Three pieces are needed—a lock arm, a lock bar, and a key bar. Cut the lock arm from a piece of pine; its outside dimensions should be $\frac{3}{4}'' \times 1'' \times 4''$. Next drill a $\frac{1}{2}''$ hole at each end so that the two holes are 3″ apart from center to center (Fig. 4). The lock bar is simply a $\frac{1}{2}''$ round dowel $5\frac{1}{2}''$ long. The key bar is a $\frac{1}{2}''$ round dowel $6\frac{1}{4}''$ long with five little key posts distributed along its length as illustrated in Figure 4. The first post is $1\frac{3}{4}''$ from the left end of the key bar and the distances between adjacent pairs of posts is 25/32″ from center to center. The posts are nothing more than $\frac{1}{4}''$ dowels glued or nailed

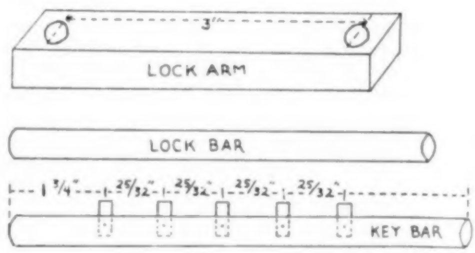

Fig. 4

into $\frac{1}{4}''$ holes drilled into the key bar. They must be arranged in a straight line and may not protrude more than 3/16″.

Assembling the lock can become a confusing trial and error process unless some definite plan is followed. By organizing the work according to the following sequence of steps no operations will need to be undone and repeated:

(1) Mark the digits 1-2-3-4-5-6-7-8-9-0 around each cylinder and note carefully which digit on each cylinder corresponds to the keyway. Assume as in Figure 5 that the keyway digits are 0-1-9-7-5.

(2) Mark an "X" on the right end piece of

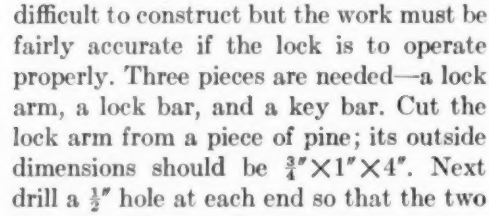

Fig. 3

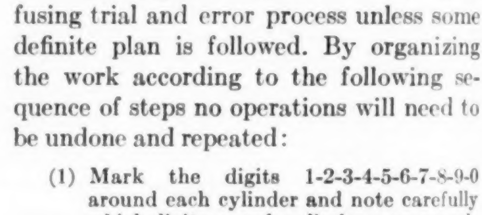

Fig. 5

the cradle and fasten it to the base with two wood screws.

(3) Slip the lock bar into one hole of the lock arm and nail it flush left at *A*.

(4) Slip the left end of the key bar through the hole in the left end piece of the cradle and then into the second hole in the lock arm. *Do not nail at B yet.*

(5) Place all cylinders on the key bar so that the open shoulders are on the right and the key posts are inside the keyways.

(6) Slip the lock bar and the key bar through the holes in the right end piece of the cradle.

(7) Turn the key bar until the numbers 0-1-9-7-5 line up on *X*. Then push the key bar to the right and nail flush left at *B*.

(8) Finally fasten the left end piece of the cradle to the base with two wood screws.

To open this particular lock set the digits 0-1-9-7-5 on *X* and pull the lock arm toward the left. The lock bar will then swing free.

By applying the Fundamental Principle of Choise it is easy to see that it is possible to set $(10)^5$ different combinations on this lock, but only one of these will open it.[2] Students will readily see that the first cylinder may be set on *X* in 10 different ways, and that for each of these ways the second cylinder may be set on *X* also in 10 different ways, etc. By continuing the explanation in this manner, students should develop a real understanding of the Fundamental Principle. It is of interest that most so-called combination locks, including the one described in this article, are not really combination locks at all, but rather "permutation" locks.

We would like to suggest that the reader turn the descriptions and drawings of this article over to his students and watch the results. Try it!

Parallel Lines Divider

This simple little device can be used to illustrate several interesting applications of the plane geometry theorem which may be stated as follows: If three or more parallels cut off equal segments on one transversal they cut off equal segments on every transversal.

The device may be constructed with narrow wooden strips cut from apple box lumber and 12 small household nails (Fig. 6). The lengths of the parallel strips, *AM*, *BL*, etc., should be 15″, and the cross strips *AF* and *MG* should be $13\frac{1}{2}''$. All parallel strips must have the same width.

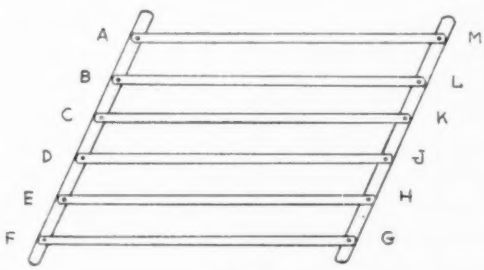

Fig. 6

To assemble the device, divide the cross strip *AF* so that $AB=BC=CD=DE=EF=2\frac{1}{2}''$, and similarly with *MG*. Then nail the strips together so that $AM=BL=CK=DJ=EH=FG=14''$. Use only one nail at each intersection. Note that the dimensions have been planned to allow for a $\frac{1}{2}''$ nailing margin at the end of each strip.

By extending or collapsing the device it is possible to divide a line segment, stick, or rod into 2, 3, 4, or 5 equal parts, provided of course that the original length of the segment, stick, or rod is not beyond the limitations imposed by the size of the device.

By adjusting the device so that the upper (or lower) edges of *AM* and *FG* coincide with the long edges of a rectangular piece of board it is possible to mark the board into 5 strips of equal width. By using either the upper or lower edges of *AM* and *DJ* the board can be divided into three strips of equal width.

The Mathematics Laboratory
Monroe High School

[2] Occasionally the Fundamental Principle of Choice is referred to as the Fundamental Theorem, or simply Fundamental Principle. It may be stated as follows: If one act can be performed in *m* different ways, and if a second act can then be performed in *n* different ways, then the two acts together can be formed in *mn* different ways, and so on for any number of acts.

NOTES ON THE HISTORY OF MATHEMATICS

Edited by VERA SANFORD
State University Teachers College, Oneonta, New York

Robert Recorde's Pasturing Problem

THE SOCIAL unrest in England that accompanied the shift from agriculture to grazing as the foreign market demanded more wool and paid better for it, was a serious situation in England in the sixteenth century. It is the basis for one of the few problems which the "Scholer" proposed to the Master in Recorde's *Grovnd of Artes* (c. 1542). Its presence in this volume makes one realize that problems of sociological and economic significance have been the concern of textbook writers even before the twentieth century.

To appreciate the timeliness of the problem in Recorde's work, it must be realized that the practice of enclosing the commons and turning arable land into sheep pasture had begun before 1500, and the discontent due to the unemployment thus caused lasted through the following century. Sir Thomas More wrote vehemently against the practice in his *Utopia* (1516). In the 1540's a member of Parliament from Wales proposed legislation not unlike Recorde's problem, and it should be noted here that Recorde himself came from the border of Wales and it is possible that the "Scholer" was suggesting something Recorde had heard himself in discussions of the situation. At any rate, the problem plans a type of social legislation that is not often connected with the Tudors. The problem reads as follows:

Scholer, And nowe doeth there come a question to my memorie whyche was demaunded of me, but I was not able to answere it, and nowe me thinketh I coulde solue it.
Mayster, Propone your question.
Scholer, There is supposed a Lawe made that for the furthering of tillage euerye manne that doeth keep sheepe, shall for euery 10 sheepe eare and sowe one acre of grounde; and for his allowance in sheepe pasture, there is appointed for euerye 4 sheepe 1 acre of pasture; Nowe is there a riche sheepemayster whiche hathe 7000 acres of grounde, and woulde gladlye keepe as many sheepe as hee mighte by that statute, I demaunde how manye sheepe shall he keepe?*

* Robert Recorde. *Grovnd of Artes, c.* 1542, 1579 ed., fol. Ff(vi).

Next to the mother tongue the language of numbers and figures is the most important symbolic possession of men. In fact it is a language within the mother tongue providing a most powerful practical and theoretical extension. In view of our present scientific and industrial conditions of life, the decay and elimination of mathematics in education is most disturbing. This default has become so common now that many persons believe that they natively lack mathematical ability. Nothing could be more crippling to the individual nor more discouraging for the future of democratic societies, if it were true. The apparent disability is due to a decay in the techniques for teaching mathematics and this in turn is due to misunderstandings of the fundamental nature and intention of mathematics.

—Catalogue of St. Johns College, Annapolis, Maryland, 1945–1946, p. 26.

BOOK SECTION

Edited by JOSEPH STIPANOWICH
Western Illinois State College, Macomb, Illinois

BOOKS RECEIVED

Elementary

Our Number Workshop 1, by Maurice L. Hartung, Henry Van Engen, and Catharine Mahoney. Paper, 97 pages, 1952. Scott Foresman and Co., 433 East Erie St., Chicago 11, Ill. $0.56.

Our Number Workshop 2, by Maurice L. Hartung, Henry Van Engen, and Catharine Mahoney. Paper, 129 pages, 1952. Scott Foresman and Co., 433 E. Erie St., Chicago 11, Ill. $0.60.

Junior High School

Arithmetic in Life and Work (Fourth Ed.) by Sidney J. Lasley and Myrtle F. Mudd, Northeast Junior High School, Kansas City, Mo. Cloth, x+259 pages, 1952. Prentice-Hall, Inc., 70 Fifth Ave., New York 11, N. Y. $1.96.

Revision Course in General Mathematics, by Clement V. Durell. Cloth, vi+125 pages +tables, 1952. G. Bell and Sons, Ltd., York House, Portugal Street, London, England. With answers, 6 shillings; without answers, 5 shillings 6 pence.

College

Algebraic Projective Geometry, by J. G. Semple, Kings College, London; and G. T. Kneebone, Bedford College, London. Cloth, vii+404 pages, 1952. Oxford University Press, 114 Fifth Ave., New York 11, N. Y. $7.00.

Practical Calculus (Second Ed.), by Claude Irwin Palmer and Claude E. Stout. Cloth, xx+470 pages, 1952. McGraw-Hill Book Co., 330 West 42nd St., New York 36, N. Y. $6.00.

Mathematics of Finance, by Edwin D. Mouzon, Jr., and Paul K. Rees. Cloth, viii+255 pages+tables, 1952. Ginn and Co., Statler Building, Boston 17, Mass. $4.60.

Numerical Methods in Engineering, by Mario G. Salvadori, and Melvin L. Baron, both of Columbia University. Cloth, xiii+258 pages, 1952. Prentice-Hall, Inc., 70 Fifth Ave., New York 11, N. Y. $5.00.

Opticks, by Sir Isaac Newton (based on the Fourth Edition, London, 1730). Paper, cxv +406 pages, 1952. Dover Publications, Inc., 1780 Broadway, New York 19, N. Y. $1.90 (Cloth, $3.95).

Miscellaneous

Trends in Production of Teaching Guides: A Survey of Courses of Study Published in 1948 Through 1950, by Eleanor Merritt and Henry Harap, George Peabody College for Teachers. Paper, 31 pages, 1952. Division of Surveys and Field Services, George Peabody College for Teachers, Nashville, Tenn. $0.50.

Introduction to Concepts and Theories in Physical Science, by Gerald Holton, Harvard University. Cloth, xviii+650 pages, 1952. Addison-Wesley Press, Inc., Cambridge 42, Mass. $6.50.

Free and Inexpensive Learning Materials, Paper, viii+194 pages, 1952. Division of Surveys and Field Services, Geoge Peabody College for Teachers, Nashville, Tennessee. $1.00.

Sargent Guide to Private Junior Colleges and Specialized Schools and Colleges. Paper, 250 pages, 1952. Porter Sargent Publishers, 11 Beacon St., Boston 8, Mass. $1.10.

The Sargent Guide to Summer Camps for Boys and Girls. Paper, 96 pages, 1952. Porter Sargent Publishers, 11 Beacon Street, Boston 8, Mass. $1.10.

REVIEWS

Growth in Arithmetic, Grade 5, John R. Clark, Harold E. Moser, and Charlotte W. Junge. New York, World Book Co., 1952. v+314 pp., $2.12.

Book 5 is one of a series of six, written for grades three through eight. Upon examining this book, one has the immediate urge to examine the rest of the series. The authors have been truly successful in providing materials that should do much to help children develop the ability to think with numbers and to promote improved methods of instruction.

One of the distinct features of this series is the pattern used in the development of a new technique or process. The plan draws upon previously learned relationships and experiences, makes definite and specific provision for discovery and encourages the formulation of generalizations, in preparation for the algorism. This same pattern is used again and again as more difficult concepts are introduced. Learning experiences such as these tend to aid in the development of confidence and result in real satisfaction in the use of numbers.

Particular emphasis is given to the development of skill in estimating and in judging the reasonableness of answers. The problem situations are drawn from the experience of children of this grade level and should hold their interest and aid in the better understanding of the use of numbers.

The format is unique, attractive, and efficiently arranged. The pages are six and one-half inches wide and this makes it possible for the book to lie flat when open. The plan, type, and general arrangement make reading and study easy. The frequent and varied types of evaluation will be a definite aid to both teacher and pupil. There is an abundance of practice material organized according to need. The functional illustrations and diagrams clarify problem situations and should do much to motivate interest.

To examine this series is a "must" on your calendar of "to do." You may challenge the rationalization used in division and contend that not all children of fifth grade level can rationalize that much. But you will want to try this plan with some of your pupils. Furthermore, you will, without doubt, agree that this series of books provides materials and suggestions that will develop insight to improved methods of teaching meaningful arithmetic.—Olive G. Wear, Fort Wayne Public Schools, Fort Wayne, Indiana.

Mathematics, a Second Course, Myron F. Rosskopf, Harold D. Aten, and William D. Reeve. New York, McGraw-Hill Book Company, 1952. xviii+365 pp., $2.80.

Mathematics, a Second Course is a plane geometry textbook with great emphasis on development of logical thinking. In fact, the organization of the book seems to be in terms of logic, with the geometry reorganized to fit the logical development. For example, some of the chapter headings are: 1. "Thinking and the Study of Geometry," 2. "Forming a Working Hypothesis," 3. "From Syllogism to Step-and-Reason in Proof," 7. "A Proposition and Its Converses," 8. "Inductive Reasoning in Geometry," 9. "Necessary and Sufficient Conditions in Proof."

There are a number of unusual features. At the end of each chapter there are two Mastery Tests, one geometric and the other nongeometric. This is part of the effort to promote "carry-over to ordinary life situations of the principles of reasoning presented in the chapter." Algebraic methods are merged with the geometry and there are frequent Algebra Refresher Exercises. The regular exercises are plentiful and are graded in difficulty to allow for individual differences without calling the student's attention to the fact. Assumptions are freely made as the need arises. Some three-dimensional geometry has been included as well as a little numerical trigonometry and some analytic geometry.

Perhaps the most remarkable feature is the "Study Assignments." There are 107 of these through the book, each stating "Given Data" (such as A and B are equidistant from C and D), "Problem? (such as to find the relationship of AB to CD)," and "Suggestions for the Proof." The suggestions are quite complete at first but gradually the student is led to discover his own conclusions, formulate a statement of them, and develop a proof. This procedure means that many of the theorems and assumptions are not stated in the book, but instead the student must write his own in a notebook. The textbook gives very careful suggestions about the notebook, and the teacher's manual which is available contains a complete list of all the assumptions, definitions, and theorems in the order of their development. We have talked much about allowing the students to discover the truths rather than throwing the truths at them. Here is a book which really allows us to do this.

As you see, this is a very unusual book. Those teachers who have the freedom, the initiative, and the ambition to teach geometry this way may find they can open up new worlds of thought for their students. The more conservative teachers would do well at least to have a copy of this book on their bookshelves for reference and to suggest an occasional new idea.—Henry Swain, New Trier Township High School, Winnetka, Illinois.

Plane Trigonometry, L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill Book Company, 1951. xii+220 pp., $3.50.

This book is a revision of the authors' well-known text and includes new theory, new proof, and new problems.

After a brief but thorough introduction to the basic trigonometric relationships, simple identities are treated extensively. The author states that "while such identities obtained are of little value in themselves, the process of expressing all parts of a figure in terms of given ones is very important." Much practice is then given involving this type of identity.

Problems involving the solution of right triangles are not introduced until Chapter 3 of the text, and only after basic relationships are firmly established. Generalized trigonometric functions are taken up in Chapter 4 which is followed by a short chapter on vectors in which the addition and subtraction of vectors and the components of a vector are treated.

Arc measure, graphs of the functions, and the conversion formulas with some introduction to trigonometric equations precede the solution of oblique triangles. The inverse functions, trigonometric equations, and complex numbers complete the list of usual topics.

Logarithms are discussed in the last chapter of the book, although their use in problem solving is introduced in Chapter 3 in connection with the solution of right triangles. Slide rule solutions are also indicated where only 3 digit accuracy is required.

At the end of each chapter there are review exercises which should prove helpful in providing continuity to the course. In general the book is well written by competent authors and the somewhat different arrangement of topics should do much to tie together the practical and theoretical aspects of trigonometry.—Herbert Hannon, Western Michigan College of Education, Kalamazoo, Michigan.

The Arithmetic of Better Business, Frank J. McMackin, John A. Marsh, and Charles E. Baten. Boston, Ginn and Company, 1951. viii+389 pp., $2.48.

This book outlines, in a very concise manner, the mathematical processes that are very frequently used in actual business practice. The summary paragraphs at the end of each section state in words that will easily be understood by the average high school student the rules to follow in working the problems.

In the hands of a teacher who is thoroughly familiar with business procedures and who could develop the reasons for performing the actual arithmetic processes effectively, this text could be a useful tool. As the book itself does not deal to any great extent with the reasons for doing the processes it outlines, it would probably prove less effective as a teaching device when used by an inadequately prepared teacher.

A ninth grade class of average ability could master the material reasonably well in two semesters. It might be more effective though, if offered for only one semester to high school seniors who had some background in general business subjects and who were interested in attaining competence in a wide variety of problems they would be likely to meet in clerical work or in operating their own small business.—JOHN W. RAU, New Trier Township High School, Winnetka, Illinois.

College Mathematics, Charles E. Clark. New York, Prentice-Hall, Inc., 1950. iv+331+46 (tables) pp., $3.85.

There are two features to this text which strike the reviewer as noteworthy: the choice of topics and the flexibility of its use. The second feature is clearly intended by the author. He states that the text can be used in one- and two-semester courses, terminal in character or preparatory to courses in analytics and calculus.

As to the topics, it should be said, first of all, that there is no departure into historical or social aspects of mathematics. It is all straight mathematics, presented with a reasonable amount of rigor. What is somewhat unusual is the lightness of treatment accorded to the manipulations of algebra and, to a lesser extent, trigonometry. There is very little of analytics. The author believes that a student learns his algebra best in the context of interesting applications to more advanced topics, such as calculus, statistics, curve fitting, and financial mathematics. As a result, the total course strikes one as rich in the kind of mathematics that one really can use in science, economics, education, etc.

While the reviewer is unable to decide whether this method is didactically successful, he thinks it provides motivation and does away with some of the dullness inherent in traditional algebra courses. Each topic is approached from a practical angle and rounded in itself; hence the book should appeal especially to the student who does not major in mathematics.

All theory is given with such brevity as would make self-study practically impossible. Sample exercises, however, are numerous and worked out in careful detail. Students' exercises are also numerous and selected to appeal to the ordinary student. In the hands of the teacher who likes the list of topics and an introduction to calculus from the point of view of approximation, it should be a useful text.—P. R. NEUREITER, State University Teachers College, Geneseo, New York.

Analytic Geometry and Calculus, William R. Longley, Percy F. Smith, and Wallace A. Wilson. Boston, Ginn and Company, 1951. ix+578 pp., $5.00.

As the title suggests, this text presents topics usually covered in introductory courses in plane and solid analytic geometry, and the differential and integral calculus.

The first two chapters deal with the fundamentals of plane analytic geometry, furnishing the background for the development of the basic concepts of the calculus theory, which first appears in Chapter 3. The elementary technique of integration is introduced on page 125.

Throughout the text, theorems and definitions are italicized. Formulas stand out clearly in bold-face type. Illustrative examples are numerous, well-chosen, and meaningful. Ample opportunity for acquiring understanding and manipulative skills is afforded by extensive lists of problems. Answers to many of the exercises can be found at the end of the text.

A collection of formulas from elementary algebra, geometry, and trigonometry, as well as tables most needed for computations is included in the last chapter.

Exposition of the mathematical theory is both lucid and reasonably rigorous. The general format of the text is excellent. The reviewer highly recommends this book to students of engineering or of the sciences, for whom early exposure to the calculus is desirable.—ARTHUR SVOBODA, De Paul University, Chicago, Illinois.

Instruments for the Enrichment of Secondary School Mathematics, Randolph Scott Gardner. Ann Arbor, Edwards Brothers, Inc., 1951. vi+98 pp., $2.50. Available from: Bookstore, New York State College, Albany New York.

What is a transit? How can I construct a slide rule? Can an abacus help me to better understand our number system? Why do we call it a protractor?

These and many other similar questions will be answered for you if you but take the time to read Professor Gardner's interesting little book. Not only should you find the information interesting in itself, but it should provide you with a supply of ammunition with which to motivate your classes.

The text is divided into four parts: (1) Measuring Instruments (Ruler, Protractor,

Vernier); (2) Calculating Instruments (Slide Rule, Abacus); (3) Miscellaneous Instruments (Pantograph, Parallel Rulers, Center Square, Parallel Dividers, Steel Square); (4) Field Instruments (Transit, Sextant, Hypsometer-Clinometer, Angle Mirror).

Each instrument is introduced by a historical study. This is followed by an explanation of the underlying mathematical theory and, in some instances, the method of construction. Then, as a clincher, the reader can find numerous exercises for solution (with their answers at the end of the chapter).

Although the author has written this book primarily for those college undergraduates preparing to teach mathematics on the secondary level, it would seem that most mathematics teachers could profit from a study of it. This may even lead to the use of more instruments in the classroom and result in the direct application of many of the principles which are too often discussed as mathematics for mathematics sake.—JOSEPH J. STIPANOWICH, Western Illinois State College, Macomb, Illinois.

Mathematics for Engineers. (Rev. Third Ed.) Raymond Dull and Richard Dull. New York, McGraw-Hill Book Company, 1951. xix+822 pp., $7.50.

This voluminous treatise on mathematics, first published in 1926, is prepared primarily as a reference book for engineers, but high school and college teachers as well will find it is a valuable acquisition for their reference shelves. The treatise is, in effect, an encyclopedia of elementary mathematics; here, under one cover, are detailed discussions of topics ranging from simple algebra to differential equations. Also included are many widely scattered topics of mathematics specially important in engineering work, such as the slide rule, vectors, empirical equations, graphical integration, etc. The treatise is not intended as a textbook; no important principles or steps are left to the reader to supply, and there are no lists of exercises for the student. The exposition is generally concise and clear, and numerous examples and graphs are included to illustrate the various discussions.

The major change in this third revised edition is the addition of two chapters: differential equations, and dimensional analysis. Other chapters have been expanded; in particular, new material has been added to the chapters on infinite series, determinants, vectors, and hyperbolic functions.—H. D. LARSEN, Albion College, Albion, Michigan.

An Outline of Statistical Methods (Fourth Ed., Rev.), College Outline Series, Herbert Arkin and Raymond R. Colton. New York, Barnes & Noble, 1950. xiv+224+47 pp., $1.50.

A publication such as this could hardly be a comprehensive treatment of statistics, but it does include a vast collection of formulas and illustrations which should meet the statistical needs of most workers in business, science, psychology, and education.

The text deals with frequency distribution and analysis, time series analysis, linear and non-linear correlation, correlation of attributes, theory of sampling, the normal curve, index numbers, graphic presentation, and a special chapter on techniques suitable for the worker in education, psychology and biology.

The material presented is clearly and concisely treated. Mathematics is deemphasized but at the close of each chapter the reader is referred to a more rigorous treatment of the material in the chapter.

The type is clear and the format interesting. Bold face and italics make reading easier. In the appendix the authors list the important formulas included in the text, a list of the most commonly used symbols in statistics, and a table of squares, square roots, cubes and cube roots.

This outline should serve the individual who wants a quick, handy reference for the simple formulas of statistics but isn't interested in a rigorous treatment of the subject matter.—IRWIN K. FEINSTEIN, Chicago Undergraduate Division, University of Illinois, Chicago, Illinois.

Some Theory of Sampling, William Edwards Deming. New York, John Wiley and Sons, Inc., 1950. xvii+602 pp. $9.00.

Understanding and facility in the use of speedy, economical, and reliable sampling techniques are essentials in the equipment of the modern statistician. It is the purpose of this book to supply these essentials by making the theory of sampling understandable and usable by those with the necessary preparation. More specifically, the author aims "to teach some theory of sampling as met in large-scale surveys in government and industry, and to develop in the student some power and desire for originality in dealing with problems of sampling."

The materials in the book are presented with sufficient elasticity as to permit it to be used as a textbook for the categories of students: (1) those in the social sciences and (2) those in the natural sciences, engineering, and industrial management. The materials for the second group should be supplemented by additional study of design of experiment and quality control from other sources.

The book has five parts. The first three are concerned with the theoretical problems which arise in planning surveys. Two applications of the theory covered in the first three parts are given in Part 4. The fifth part is devoted to advanced statistical theory and is intended to provide enough knowledge to enable the student to develop new techniques in sampling. Part 1 is devoted to a consideration of the preliminary problems and plans of a large scale survey, including the specification of the precision to be aimed at the survey and the possible sources of error. The author's illuminating discussion of the "Steps in Taking a Survey" does much to justify the need for the theory developed in this and the next two parts.

Part 2 develops the theory relative to designing the sample mathematically, after the preliminary plans have been laid, to achieve the desired precision at the lowest cost. The theory is introduced through a discussion of moments and expected values. Then the author discusses the procedures of selecting random samples and develops variances of statistics from random samples. The theory relative to variances is then used in determining the required size and cost of a sample that will yield the required precision. While single-stage sampling provides the simplest theory, a required precision may sometimes be met cheaper by multi-stage sampling. The presentation of the theory of multi-stage sampling is followed by sections on ratio-estimates and cluster sampling. The next chapter, "Allocation in Stratified Sampling," discusses the problem of the most efficient distribution of a sample amongst the several strata into which a universe may have been divided, in order to achieve maximum precision per unit cost for estimated characteristics of the universe. The distinction between enumerative and analytic studies is brought out in the following chapter. The final chapter in this part introduces some applications of the theory of probability to the control of risks that are involved in formulating rules of acceptance based on samples, particularly in the inspection of industrial product.

The appraisal of the results after the survey has been finished, to see what the results mean and how the next survey can be improved, is discussed in Part 3. Consideration is given to problems connected with estimates of various statistical characteristics of the universe, the reliability of the estimates, and the interpretation of the estimates.

Part 4 presents two applications of the materials covered in the preceding chapters. The first application is an estimation of tire inventories based on a sample survey of tire dealers. The second application is concerned with a population sample for Greece. These applications were described quite completely in the *Journal of the American Statistical Association* in 1946 and 1947, respectively.

Although the author disclaims any intention for use of the book as a textbook in mathematical statistics, the discussion of moments, expected values, and the derivation of sampling formulas in chapters 3–10, together with the inclusion in Part 5 of a great deal of the statistical theory on such topics as the binomial and related distributions, Gamma and Beta functions and distribution theory, would seem to indicate that the author is too modest.

The topics in the book are arranged in logical and systematic order and the materials are treated in a competent and professional manner. Students are assumed to have had some training in the calculus for mastery of certain sections. The clever selection of numerous exercises and remarks which skillfully develop and extend the main discussion is one of the unusual features.

In summary, Dr. Deming's excellent work is one which every person in any way interested in surveys and sampling will find useful and stimulating.—Joseph A. Pierce, Texas Southern University, Houston, Texas.

Experimental Designs, William G. Cochran and Gertrude M. Cox. New York, John Wiley and Sons, Inc., 1950. ix+454 pp., $5.75.

This book is very well organized and written in a clear, concise manner to make it appealing to anyone interested in the field of statistics or experimental design. The first three chapters present a classic discussion of the objects, construction, and conclusions to be drawn from experimental design. The initial stages of the problem; what data to use, what techniques, and in particular certain pitfalls to be avoided, all are thoroughly discussed. The remaining twelve chapters are concerned with a detailed study of the first three chapters as applied to over 150 experimental designs.

This book should certainly be in the library of every statistician—William E. Felling, St. Louis University, Parks College of Aeronautical Technology, East St. Louis, Illinois.

Statistical Decision Functions, Abraham Wald. New York, John Wiley and Sons, 1950. ix+179 pp., $5.00.

This is a very advanced book presenting original research which the author has done over a period of years. Before his untimely death in an airplane crash, Professor Wald was one of the world's leading authorities on theoretical statistics. This book is designed as an exposition of his work on statistical decision functions; with their use he was able to develop a statistical theory which can be applied to much more general situations than the previous theory.

His methods involve the free use of such subjects as measure and integration theory, set theory, topology, etc. There is also a close relationship with von Neumann's theory of games. Thus, unavoidably, this book is accessible only to the advanced mathematician.—M. Gaffney, Northwestern University, Evanston, Illinois.

The Elements of Mathematical Analysis (Second Ed.), 2 vols., J. H. Mitchell and M. H. Belz. London, Macmillan and Co., 1952. xxiii+xi+1087 pp., $13.60 the set.

This set of two volumes is essentially a single book bound in two parts. The pagination runs through the two volumes, the complete table of contents is in the first volume and the complete index in the second. The authors state that they endeavored to make this book conform to three main conditions: (1) "The book should assume as known only those elements of algebra, geometry and trigonometry which are taught in secondary schools to all those preparing to attend any lectures in mathematics at a univeristy." (2) "The book should form a practical or working text provided with an abundance of illustrative examples treated at length and of other examples to be solved by the student."

(3) "The subject should be expounded on the basis of the theory of real numbers, geometrical notions being employed only illustratively and not as replacing abstract discussions. In other words, the language and the methods of demonstration were to be those of the advanced treatises so far as was at all possible." The authors (professors at the University of Melbourne) assume under (1) that "these elements in practice include some rudiments of differential calculus." They also assume that the student is familiar with mathematical induction, graphs of conic sections in standard positions, graphs of the six trigonometric functions, the relation sin $x<|x|$, . . . The contrast in the secondary school preparation of prospective college students in this country and in Australia is also evident in the mathematical maturity expected of the reader throughout the book.

The book includes a detailed discussion of differential and integral calculus with related topics on functions of real variables. The terminology is precise but differs noticeably from that in most of our texts as, for example, in the use of upper barrier for upper bound, one-way function for monotonic function, and dimetric function for function of two variables. The book is aptly described by the above three purposes of the authors and may be effectively used to develop an understanding of the elements of mathematical analysis.—B. E. MESERVE, University of Illinois, Urbana, Illinois.

Foundations of Analysis, Edmund Landau (Translated from the German by F. Steinhardt). New York, Chelsea Publishing Company, 1951. xiv+134 pp., \$3.25.

In the course of their mathematical studies, many college students do not have the occasion to question the logical foundations of the number system. In this very clear book Landau presents a short and direct development of the number system from the Peano axioms in a manner that requires no special previous knowledge of higher mathematics but only a logical mind and the ability to grasp a few abstract concepts used in the beginning of the book. The author develops the arithmetical properties of the integers, the rational numbers, the irrational numbers (according to Dedekind), and the complex numbers, putting in all details, even at the expense of being repetitive in places where he could have said "by an argument similar to that previously used." This facilitates the reading by leaving little to be filled in by the reader. This edition is a rather literal translation of the original German edition, completely preserving the typical "Landau style" of "definition," "theorem," "proof."—GEORGE SPRINGER, Northwestern University, Evanston, Illinois.

Numerical Solutions of Differential Equations (First American Ed.), H. Levy and E. A. Baggott. (Published in England under the title Numerical Studies in Differential Equations.) New York, Dover Publications, 1950. viii+238 pp., \$3.00.

This treatise will be an excellent supplement for the research worker in applied mathematics. It has a complete resume on the graphical solution of differential equations and contains methods of numerical integration over both limited and wide ranges for first, second and higher order differential equations. The emphasis in all cases lies with accuracy in the final result in such methods as those of Euler, Runge, and Kutta [and introduces very simply the method of forward integration for wide range problems].

While there are many illustrative examples in the work, the number of problems is rather restricted for use as a text. A more complete bibliography would enhance the book.—EMIL J. WALCEK, Parks College of Aeronautical Technology, East St. Louis, Illinois.

Tensor Analysis, Theory and Applications, I. S. Sokolnikoff. New York, John Wiley and Sons, Inc., 1951. ix+335 pp., \$6.00.

This book is designed for a graduate course in tensor analysis and is aimed principally at applied mathematicians. The machinery of tensors is developed at the beginning before any applications are made. Although this has a slight tendency to make some of the concepts appear unmotivated, nevertheless there is a great advantage in making clear how much of the development in the later applications is pure tensor calculus; it also enhances the value of the book as a reference work.

The first chapter, on vector spaces and matrices, lays great stress on the concept of invariance, the understanding of which is essential in the mathematical description of nature. This idea permeates, as it should, the whole book. Chapter 3 is devoted to the differential geometry of curves and surfaces in 3-space and includes a brief discussion of Riemannian manifolds. In Chapter 4 the fundamental principles of analytical mechanics are well-treated and, in particular, the equations of Lagrange and Hamilton are derived. The last two chapters are concerned respectively with relativity theory and the mechanics of continuous media. The author has provided references for theorems stated without proof. Throughout the book there are many splendid discussions on historical background which greatly enhance the pleasure of reading it.—W. E. JENNER, Northwestern University, Evanston, Illinois.

Célèbres Problèmes Mathématiques. Edouard Callandreau. Paris, Editions Albin Michel, 1949 477 pp.

The book by Callandreau is probably written with the aim to fill the void created by the absence of a collection of the most important problems in mathematics which border on the elementary phases of the subject. If this were the author's aim, he certainly succeeded in accomplishing a Herculean task. He not only presented all important and fundamental problems in various fields of mathematics, but he

(*Continued on page 555*)

MATHEMATICAL RECREATIONS

Edited by Aaron Bakst
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Calendar problems have fascinated many a mathematician. Every now and then a formula for the determination of the days of the week or of the Easter date is published. Such formulas are valid for a certain number of years. The reason for such limitations is associated with the fact that the length of the year, that is, the length of the period of the year is not the same. The earth completes one cycle on its orbit around the sun in approximately 365.24220 days. Furthermore, the length of the year is not constant. Due to friction which is caused by the gravitational attraction of the moon, the planets, and, perhaps other celestial bodies, the length of the year is gradually decreasing. If this decrease is not perceptible, it still cannot be disregarded. All these factors, when taken into consideration, complicate all calendar computations. Anyone who sets out to devise a *perpetual calendar*, that is, an instrument which would enable one to determine the days of the week from here to eternity, should make arrangements to live forever. However, calendars which are valid for several centuries are possible, and many of them were constructed. Below is a "perpetual calendar" which is good until the year 2700 A.D.[1]

Several types of "perpetual calendars" have been published during the last few decades. Some of them are tabular, others are in the form of graphs.[2] However, the "Schubert Calendar" is, in the opinion of this department, the most satisfactory from the point of view of the simplicity of construction and operation.

One may question the wisdom of classifying the problem of a "perpetual calendar" as a mathematical recreation. On the surface such an instrument may seem to be a curiosity. However, a closer examination of its potentialities as an instrument and a device in a classroom situation will reveal certain facts which might escape us at a first glance. Among the objectives of algebra-instruction there stands out a universally accepted aim of developing the ability to read and interpret tabular data. Generally, the tabular data which is presented to the pupils is such which, by and large, rarely arouses genuine enthusiasm. Furthermore, the successful transfer of the abilities "to read" data from tabulated statistical information to tables of mathematical functions is extremely doubtful. Tables of logarithmic and trigonometric functions are usually so arranged that their reading requires the location of the intersections of columns and rows. Such a technique is more difficult than the direct reading from a two adjacent columns of an argument and a function.

A "perpetual calendar" may appeal to and arouse a genuine interest of the pupils because its reading yields information which is difficult to secure by any other means. Moreover, such information may contain characteristics of "personal" nature which may concern the individual pupil as well as groups of pupils. Consider the following questions:

(1) Jack was born on August 23, 1938. What day of the week was it?

(2) Columbus discovered America on October 12, 1492. What day of the week was it?

(3) On what day of the week was the Declaration of Independence signed?

[1] This calendar was devised by Dr. Hermann Schubert. See: H. Schubert, *Mathematische Mussenstunden*, Leipzig: Göschen'sche Verlagshandlung, 1900, Vol. 2, pp. 81–88.

[2] See Bibliography.

PERPETUAL CALENDAR

Old Style Centuries			I						
0 . .	7 . .	14 . .	c	d	e	f	g	a	b
1 . .	8 . .	15 . .	b	c	d	e	f	g	a
2 . .	9 . .	16 . .	a	b	c	d	e	f	g
3 . .	10 . .	17 . .	g	a	b	c	d	e	f
4 . .	11 . .	18 . .	f	g	a	b	c	d	e
5 . .	12 . .	19 . .	e	f	g	a	b	c	d
6 . .	13 . .	20 . .	d	e	f	g	a	b	c
Years in Centuries			00, 06, 17, 23, **28**, 34, 45, 51, **56**, 62, 73, 79, **84**, 90	01, 07, **12**, 18, 29, 35, **40**, 46, 57, 63, **68**, 74, 85, 91, **96**	02, 13, 19, **24**, 30, 41, 47, **52**, 58, 69, 75, **80**, 86, 97	03, **08**, 14, 25, 31, **36**, 42, 53, 59, **64**, 70, 81, 87, **92**, 98	09, 15, **20**, 26, 37, 43, **48**, 54, 65, 71, **76**, 82, 93, 99	**04**, 10, 21, 27, **32**, 38, 49, 55, **60**, 66, 77, 83, **88**, 94	05, 11, **16**, 22, 33, 39, **44**, 50, 61, 67, **72**, 78, 89, 95

Remarks

1. The numbers in heavy type [56, 64] indicate leap years in the New Style.
2. Years divisible by 400 are leap years in the New Style.
3. The first day of the New Style accepted by the Catholic Church—October 15, 1582.
4. The First day of the New Style accepted in Protestant Germany, March 1, 1700.

New Style Centuries			I						
26 . .	22 . .	18 . .	a	b	c	d	e	f	g
25 . .	21 . .	17 . .	c	d	e	f	g	a	b
24 . .	20 . .	16 . .	e	f	g	a	b	c	d
23 . .	19 . .	15 . .	f	g	a	b	c	d	e

II							
LEAP YEAR JANUARY	g	a	b	c	d	e	f
LEAP YEAR FEBRUARY	d	e	f	g	a	b	c
January	f	g	a	b	c	d	e
February	c	d	e	f	g	a	b
March	c	d	e	f	g	a	b
April	g	a	b	c	d	e	f
May	e	f	g	a	b	c	d
June	b	c	d	e	f	g	a
July	g	a	b	c	d	e	f
August	d	e	f	g	a	b	c
September	a	b	c	d	e	f	g
October	f	g	a	b	c	d	e
November	c	d	e	f	g	a	b
December	a	b	c	d	e	f	g
1. 8. 15. 22. 29.	M	Tu	W	Th	F	Sa	S
2. 9. 16. 23. 30.	Tu	W	Th	F	Sa	S	M
3. 10. 17. 24. 31.	W	Th	F	Sa	S	M	Tu
4. 11. 18. 25.	Th	F	Sa	S	M	Tu	W
5. 12. 19. 26.	F	Sa	S	M	Tu	W	Th
6. 13. 20. 27.	Sa	S	M	Tu	W	Th	F
7. 14. 21. 28.	S	M	Tu	W	Th	F	Sa

Abbreviations

M = Monday, Tu = Tueday, W = Wednesday
Th = Thursday, F = Friday, Sa = Saturday
S = Sunday.

Instructions

1. Locate in Table I the letter which corresponds to the intersection of the row of Centuries and the column of the Years.
2. Locate this letter in the row of the month of Table II.
3. Follow the column of this letter to the intersection of the row with the given date of the month. The intersection will give the day of the week.

(4) On what day of the week will vacation begin in 1953?
(5) On what day of the week was President Lincoln assassinated?

There are many other questions which may be posed. As a matter of good procedure it would be advisable to permit the pupils to formulate their own questions.

Instructions for the use of the perpetual calendar are printed in the lower right-hand corner. Thus, for example, in order to ascertain the day of the week on which Columbus discovered America, that is, October 12, 1492, we proceed as follows. We note that this was an Old Style date. The century is 14. We locate the year 92. The intersection of the row of 14 with the column of 92 yields the letter "*f*." In Table II we locate the position of the letter "*f*" in the line of the month "October." The intersection of the column of the letter "*f*" with the line of the number 12 gives the letter "*F*." Thus, Columbus discovered America on Friday.

On what day will Christmas of 1952 fall? This is a New Style date. The intersection of the column of the number 52 with the row of the number 19 yields the letter "*a*." In Table II we locate the letter "*a*" in the line of the month "December." The intersection of the column of the letter "*a*" with the line of the number 25 gives the letters "*Th*." Thus, Christmas Day in 1952 will fall on Thursday.

This calendar may be employed in reverse order. For example, we may ascertain the date of Labor Day in 1953. We proceed as follows. In Table I we find that the intersection of the column of 53 with the row of 19 gives the letter "*b*." Table II indicates that the column of the letter "*b*" in the line of the month "September" contains the letter "*M*" in the line of the dates 7, 14, 21, and 28. Labor Day is the first Monday in September. Thus, in 1953 Labor Day will fall on September 7th.

Bibliography

1. Couderc, P. *Le Calendrier*. Paris: Presses Universitaires de France, 1948.
2. Kraitchik, M. *La Mathématique des Jeux*. Bruxelles, 1930. Contains a perpetual calendar nomogram and a perpetual calendar circular chart.
3. Schubert, H. *Mathematische Mussenstunden*. Leipzig: G. J. Göschen'sche Verlagshandlung, 1900, Vol. 2.
4. *The World Almanac*. New York: New York World Telegram and Sun, 1952, p. 785.
5. *Information Please Almanac*. New York: Harper's, 1952.

Books

(*Continued from page 552*)

supplied ample historical notes and other commentaries which are extremely valuable and useful in the study of these problems.

The divisions of the book are: Arithmetic; Algebra and Analysis; Plane and Solid Geometry; Analytic Geometry; Rational Mechanics; Celestial Mechanics, Astronomy, and Navigation; Maxima and Minima.

The problems are arranged in order of their difficulty as well as (if it were feasible) in some historical order. The commentaries are replete with references and quotations from subsequent works. In other words, further developments of the problems are clearly indicated and appropriately presented. Whenever it was feasible, the author simplified the presentation so as to reach a larger reading audience.

The list of the problems includes a variety of topics. Most of these would be extremely useful in mathematics clubs on the college level. However, some of these problems could be further simplified and brought down to the level of a high school mathematics club. For example, the problem of "systems of numeration" which states that (a) any integer is decomposable into a sum of powers of any integer $B>1$ so that no power is taken more than $(B-1)$ times, and (b) in all systems of numeration (taken at some integral base) the double of the base less than one $(B-1)$, if written in reverse order, is equal to the square of the base less than 1. In other words, $2(B-1)$ written in reverse order is equal to $(B-1)^2$. For example, in the decimal system of numeration $2\cdot 9=18$ and $9^2=81$.

There is only one other book which approaches the work of Callandreau. This was published in Germany in 1933. Reference is made to *Triumph der Mathematik* by Heinrich Doerrie (Ferdinand Hirt, Breslau). Many of the problems which are given by Callandreau appear in this book. However, the work of Doerrie lacks commentaries and any additional references.

Unfortunately there isn't any work in the English language which would present to the mathematical reading public works of this type. Perhaps some venturesome publisher will find it possible to publish a work of this type. Such a book in English is really needed by every teacher of mathematics.—Aaron Bakst, Flushing, New York.